19.1.10

Maß- u. Wahrscheinlichkeitsth. UE

XII,

1) X_1, X_2, X_3 i.i.d $\quad X_i \sim U_{0,1} \Rightarrow f_{X_i} = 1\!\!1_{[0,1]}$

ges.: Verteilung von $X_1+X_2+X_3$

$$f_{X_1+X_2}(x) = \int_{\mathbb{R}} f_{X_1}(t)\, f_{X_2}(x-t)\, d\lambda(t)$$

$$= \int_{\mathbb{R}} 1\!\!1_{[0,1]}(t) \cdot 1\!\!1_{[0,1]}(x-t)\, d\lambda(t)$$

$$\overset{\substack{x-t\in[0,1] \\ \Leftrightarrow\ t\in[x-1,x]}}{=} \int_{\mathbb{R}} 1\!\!1_{[0,1]}(t) \cdot 1\!\!1_{[x-1,x]}(t)\, d\lambda(t) = \int_{\mathbb{R}} 1\!\!1_{[0,1]\cap[x-1,x]}(t)\, d\lambda(t)$$

$$\Rightarrow f_{X_1+X_2}(x) = \begin{cases} \int_{\mathbb{R}} 1\!\!1_{[0,x]}(t)\, d\lambda(t) = \lambda([0,x]) = x & 0 \le x \le 1 \\ \int_{\mathbb{R}} 1\!\!1_{[x-1,1]}(t)\, d\lambda(t) = \lambda([x-1,1]) = 2-x & 1 < x \le 2 \\ 0 & \text{sonst} \end{cases}$$

$$f_{X_1+X_2+X_3}(x) = f_{X_1+X_2} * f_{X_3}(x) = \int_{\mathbb{R}} f_{X_1+X_2}(t)\, f_{X_3}(x-t)\, d\lambda(t)$$

$$= \int_{\mathbb{R}} \left(1\!\!1_{[0,1]}(t)\cdot t + 1\!\!1_{(1,2]}(t)\,(2-t)\right) \underbrace{1\!\!1_{[0,1]}(x-t)}_{=1\!\!1_{[x-1,x]}(t)}\, d\lambda(t)$$

$$= \int_{\mathbb{R}} t \cdot 1\!\!1_{[0,1]\cap[x-1,x]}(t)\, d\lambda(t) + \int_{\mathbb{R}} (2-t)\, 1\!\!1_{(1,2]\cap[x-1,x]}(t)\, d\lambda(t)$$

$0 \le x \le 1$: $\quad = \int_{\mathbb{R}} t\, 1\!\!1_{[0,x]}(t)\, d\lambda(t) = \int_0^x t\, dt = \frac{x^2}{2}$

$1 < x \le 2$: $\quad = \int_{\mathbb{R}} t\, 1\!\!1_{[x-1,1]}(t)\, d\lambda(t) + \int_{\mathbb{R}} (2-t)\, 1\!\!1_{[1,x]}(t)\, d\lambda(t)$

$$= \int_{x-1}^{1} t\, dt + \int_1^x (2-t)\, dt = \frac{t^2}{2}\Big|_{x-1}^{1} + \left(2t - \frac{t^2}{2}\right)\Big|_1^x = 3x - x^2 - \frac{3}{2}$$

$2 < x \le 3$: $\quad = \int_{\mathbb{R}} (2-t)\, 1\!\!1_{[x-1,2]}\, d\lambda(t) = \int_{x-1}^{2} (2-t)\, dt = \left(2t - \frac{t^2}{2}\right)\Big|_{x-1}^{2} = \frac{(x-3)^2}{2}.$

$$\Rightarrow f_{X_1+X_2+X_3}(x) = \begin{cases} \frac{x^2}{2} & x\in[0,1] \\ 3x - x^2 - \frac{3}{2} & x \in (1,2] \\ \frac{(x-3)^2}{2} & x\in(2,3] \\ 0 & \text{sonst} \end{cases}$$

2) $X_1, \dots, X_n$ i.i.d $\qquad X_i \sim E_{x_\tau} = \gamma(1,\tau)$

$S_n := X_1 + \dots + X_n$

a) Bestimme f_{S_n}

$$X_i \sim \gamma(1,\tau) \Rightarrow S_n = \sum_{i=1}^n X_i \sim \gamma(n,\tau) \qquad \text{lt. Additionstheorem}$$

$$\Rightarrow f_{S_n}(x) = \frac{\tau^n}{\Gamma(n)} x^{n-1} e^{-\tau x} = \frac{\tau^n}{(n-1)!} x^{n-1} e^{-\tau x} \qquad (x \geq 0)$$

b)
$$P([S_n > t]) = \int_4^\infty f_{S_n}(x)\,dx = \int_4^\infty \frac{\tau^n}{(n-1)!} \underbrace{x^{n-1}}_{g} \underbrace{e^{-\tau x}}_{f'}$$

$$= \frac{\tau^n}{(n-1)!} \frac{x^{n-1}}{-\tau} e^{-\tau x} \Big|_4^\infty + \int_4^\infty \frac{\tau^n}{(n-1)!} \frac{n-1}{\tau} x^{n-2} e^{-\tau x}\,dx$$

$$= \frac{\tau^{n-1}}{(n-1)!} 4^{n-1} e^{-\tau 4} + \underbrace{\int_4^\infty \frac{\tau^{n-1}}{(n-2)!} x^{n-2} e^{-\tau x}\,dx}_{P([S_{n-1} > t])}$$

$$= \sum_{k=0}^{n-1} \frac{\tau^k}{k!} 4^k \cdot e^{-\tau 4}$$

c) N_t ... Anzahl Reparaturen in $[0,4]$

$$P([N_4 = n]) = P([S_n \leq 4 < S_{n+1}])$$

$$= P([S_{n+1} > 4]) - P([S_n > 4])$$

$$= \frac{\tau^n}{n!} 4^n e^{-\tau 4} = \frac{(\tau 4)^n}{n!} e^{-\tau 4}$$

$$\Rightarrow N_4 \sim P_{\tau 4}.$$

3) a) ZZ: ν sign. Maß auf $(\Omega, \mathfrak{S})$

$$\Rightarrow |\nu|(A) = \sup\left\{ \sum_{i=1}^n |\nu(A_i)| \;\Big|\; \bigcup_{i=1}^n A_i \subseteq A \wedge A_i \cap A_j = \emptyset \;\forall i \neq j \right\} =: S$$

„$\leq$" Sei P, N Hahn-Zerlegung von Ω bzgl. ν, d.h.

$$|\nu|(A) = |\nu|((A \cap P) \cup (A \cap N)) = |\nu|(\underbrace{A \cap P}_{=:A_1}) + |\nu|(\underbrace{A \cap N}_{=:A_2})$$

$$= \nu(A_1) - \nu(A_2)$$

$$= |\nu(A_1)| + |\nu(A_2)| \leq S$$

„$\geq$" $\forall A_1, \ldots, A_n$ mit $A_i \cap A_j = \emptyset \ \forall i \neq j \ \wedge \ \bigcup_{i=1}^n A_i \subseteq A$ gilt:

$$\sum_{i=1}^n |\nu(A_i)| = \sum_{i=1}^n |\nu^+(A_i) - \nu^-(A_i)|$$
$$\leq \sum_{i=1}^n |\nu^+(A_i)| + |\nu^-(A_i)|$$
$$\overset{\text{Add.}}{=} \nu^+\left(\bigcup_{i=1}^n A_i\right) + \nu^-\left(\bigcup_{i=1}^n A_i\right)$$
$$\overset{\text{Sub-Add.}}{\leq} \nu^+(A) + \nu^-(A) = |\nu|(A)$$

$\Rightarrow S \leq |\nu|(A).$

b) ~~$|\mu+\nu(A_i)|$~~ ZZ: $|\mu+\nu| \leq |\mu| + |\nu|$

$$|\mu+\nu|(A) = \sup\left\{\sum |(\mu+\nu)(A_i)| \ldots\right\}$$
$$\leq \sup\left\{\sum |\mu(A_i)| + |\nu(A_i)| \,\Big|\, \ldots\right\}$$
$$\leq \sup\left\{\sum |\mu(A_i)| \,\Big|\, \ldots\right\} + \sup\left\{\sum |\nu(A_i)| \,\Big|\, \ldots\right\}$$
$$= |\mu|(A) + |\nu|(A).$$

4) $\varrho(A) := \begin{cases} |A| & |A| < \infty \\ -|A^c| & |A^c| < \infty \end{cases}$ $\qquad \mathcal{A} = \{A \subseteq \mathbb{R} \mid |A| < \infty \vee |A^c| < \infty\}$

(i) ZZ: ϱ ist σ-additiv auf $\mathcal{A}$, d.h.

$$\varrho\left(\bigcup_i A_i\right) = \sum_i \varrho(A_i) \text{ für } A_i \cap A_j = \emptyset \ \forall i \neq j \ \wedge \ A := \bigcup_i A_i \in \mathcal{A}.$$

<u>1. Fall:</u> $|A| < \infty \Rightarrow A_i = \emptyset$ für fast alle $i \in \mathbb{N} \wedge |A_i| < \infty \ \forall i \in \mathbb{N}$

$$\Rightarrow \varrho(A) = |A| \underset{A_i \text{ disj.}}{=} \sum_i |A_i| = \sum_i \varrho(A_i)$$

<u>2. Fall:</u> $|A| = \infty$

- $|A_i| < \infty \quad \forall i \in \mathbb{N}$

$\Rightarrow |A| = \aleph_0 \Rightarrow |A^c| > \aleph_0 \Rightarrow |A| = |A^c| = \infty \Rightarrow A \notin \mathcal{A}.$

- $\exists j \in \mathbb{N}$ mit $|A_j| = \infty \Rightarrow |A_j^c| < \infty$

$A_i \cap A_j = \emptyset \ \forall i \neq j \Rightarrow A_i \subseteq A_j^c \ \forall i \in \mathbb{N}$

$$\Rightarrow \bigcup_{i \neq j} A_i \subseteq A_j^c \Rightarrow \left|\bigcup_{i \neq j} A_i\right| \leq |A_j^c| < \infty$$

$$A^c = \bigcap_i A_i^c = A_j^c \setminus \bigcup_{i \neq j} A_i$$

$$\Rightarrow |A^c| = |A_j^c| - \sum_{i \neq j} |A_i| \Leftrightarrow \underbrace{-|A^c|}_{\varrho(A)} = \underbrace{-|A_j^c|}_{\varrho(A_j)} + \sum_{i \neq j} |A_i|$$

$$\varrho(A) = \varrho(A_j) + \sum_{i \neq j} \varrho(A_i) = \sum_{i} \varrho(A_i).$$

(ii) ZZ: ρ kann nicht zu signiertem Maß auf $\mathcal{A}_\sigma(\mathcal{A})$ fortges. werden.

$$\rho(\mathbb{N}) = \rho\left(\bigcup_n \{n\}\right) = \sum_n 1 = \infty$$

$$\rho(\mathbb{R}\setminus\mathbb{N}) = \rho(\mathbb{R}) - \rho(\mathbb{N}) = 0 - \infty = -\infty$$

$$\Rightarrow \rho : \mathcal{A}_\sigma(\mathcal{A}) \to [-\infty, \infty] \Rightarrow \rho \text{ ist kein sign. Maß.}$$

6) $(\Omega, \mathcal{S}, \mu)$ endlicher Maßraum, $\mu(\Omega) > 0$

ν endliches Maß, $\nu = \nu_c + \nu_s$ mit $\nu_c \ll \mu \wedge \nu_s \perp \mu$

$\mathcal{A}$ Teilsigmaalgebra von $\mathcal{S}$, $\tilde{\nu} := \nu|_\mathcal{A}$ Restriktion von ν auf $\mathcal{A}$

$\tilde{\nu} = \tilde{\nu}_c + \tilde{\nu}_s$ mit $\tilde{\nu}_c \ll \mu|_\mathcal{A}$, $\tilde{\nu}_s \perp \mu|_\mathcal{A}$

ZZ: $\tilde{\nu}_c(A) \geq \nu_c(A) \wedge \tilde{\nu}_s(A) \leq \nu_s(A) \quad \forall A \in \mathcal{A}$.

Sei $N \in \mathcal{A}$ mit $\mu|_\mathcal{A}(N) = \mu(N) = 0 \Rightarrow \tilde{\nu}_c(N) = 0$

$\tilde{\nu}_s(N^c) = 0$

$$\tilde{\nu}_c(A) = \tilde{\nu}_c(A \cap N^c) = \tilde{\nu}_c(A \cap N^c) + \tilde{\nu}_s(A \cap N^c) = \tilde{\nu}(A \cap N^c)$$

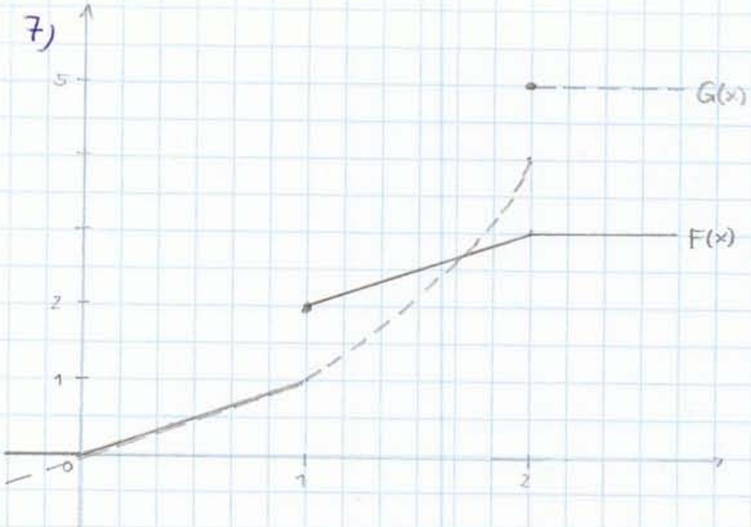
$$= \nu(A \cap N^c) = \nu_c(A \cap N^c) + \nu_s(A \cap N^c)$$

$$= \nu_c(A \cap N) + \nu_c(A \cap N^c) + \nu_s(A \cap N^c)$$

$$= \nu_c(A) + \nu_s(A \cap N^c) \geq \nu_c(A). \qquad \forall A \in \mathcal{A}.$$

$$\tilde{\nu}_s(A) = \tilde{\nu}(A) - \tilde{\nu}_c(A) \leq \nu(A) - \nu_c(A) = \nu_s(A) \qquad \forall A \in \mathcal{A}.$$

7)



$$\frac{d\mu_{G_c}}{d\mu_{F_c}} = \frac{\frac{d\mu_{G_c}}{d\lambda}}{\frac{d\mu_{F_c}}{d\lambda}} = \frac{G'}{F'} = \begin{cases} \frac{1}{1} = 1 & x \in (0,1) \\ \frac{2x}{1} = 2x & x \in (1,2) \end{cases}$$

$$\frac{d\mu_{G_c}}{d\mu_{F_c}} = \frac{d\mu_{G_c}}{d\lambda} \Rightarrow G_c(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ x^2 & 1 \leq x < 2 \\ 4 & x \geq 2 \end{cases}$$

$$\Rightarrow G_s(x) = G(x) = G_c(x) = \begin{cases} x & x \leq 0 \\ 0 & 0 < x < 2 \\ 1 & x \geq 2 \end{cases}$$