12.1.10

Maß- u. Wahrscheinlichkeitsth. UE

XI)

1) $f_X(x) = \frac{a}{\pi(a^2+x^2)}$ $(a>0)$, $Y := X^n$ $(n \in \mathbb{N})$, ges.: f_Y

$Y = T \circ X$ für $T: x \to x^n$

T ist streng monoton auf $(-\infty, 0)$ bzw. $[0,\infty)$

$G_1(y) := (T|_{(-\infty,0)})^{-1}(y) = -\sqrt[n]{|y|}$

$G_2(y) := (T|_{[0,\infty)})^{-1}(y) = \sqrt[n]{y}$ $\qquad (y \geq 0$ für n gerade$)$

$$\overset{9.74}{\Rightarrow} f_Y(y) = f_X(G_1(y))\,|G_1'(y)| + f_X(G_2(y))\,|G_2'(y)|$$

$$= \begin{cases} \frac{a}{\pi(a^2+y^{\frac{2}{n}})} \cdot \frac{1}{n} y^{\frac{1}{n}-1} & n \text{ unger.} \\ \frac{2a}{\pi(a^2+y^{\frac{2}{n}})} \cdot \frac{1}{n} y^{\frac{1}{n}-1} \cdot \mathbb{1}_{[0,\infty)} & n \text{ ger.} \end{cases}$$

2) X_1, X_2 i.i.d. mit $X_i \sim N(0,1)$

$Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$, ges.: f_{Y_1}, f_{Y_2}

- $f_{Y_1}(x) = P([Y_1 = x]) = P([X_1 + X_2 = x]) = P([X_1 = x-y, X_2 = y]) = P([X_1 = x-y]) \cdot P([X_2 = y])$

$$= \int_{\mathbb{R}} f(x-y)\, f(y)\, d\lambda(y)$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}[(x-y)^2+y^2]}\, dy$$

$$= \frac{1}{\sqrt{2\pi}\cdot\sqrt{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\frac{1}{\sqrt{2}}}\, e^{-\frac{1}{2}\left(\frac{x}{\sqrt{2}}\right)^2} e^{-\frac{1}{2}\left(\frac{y-\frac{x}{2}}{\frac{1}{\sqrt{2}}}\right)^2} dy$$

$$= \frac{1}{\sqrt{2\pi}\cdot\sqrt{2}}\, e^{-\frac{1}{2}\left(\frac{x}{\sqrt{2}}\right)^2}$$

$\Rightarrow Y_1 \sim N(0,2)$

- $f_{Y_2}(x) = P([X_1 = x-y, X_2 = -y]) = P([X_1 = x-y]) \cdot \underbrace{P([X_2 = -y])}_{= P([X_2 = y])}$

$\Rightarrow Y_2 \sim N(0,2)$

3) $(\mathbb{N}^2, \mathcal{P}(\mathbb{N}^2), \zeta\otimes\zeta)$

$$f(n,m) = \begin{cases} 2-2^{-n} & n=m \\ -2+2^{-n} & n=m+1 \end{cases} \Rightarrow f(n,m) = (2-2^{-n})\,\mathbb{1}_{\{m\}}(n) + (-2+2^{-n})\,\mathbb{1}_{\{m+1\}}(n)$$

$$\int_{\mathbb{N}}\left[\int_{\mathbb{N}} f(n,m)\, d\zeta(n)\right] d\zeta(m) = \int_{\mathbb{N}} (2-2^{-m})\,\zeta(\{m\}) + (-2+2^{-m+1})\,\zeta(\{m+1\})\; d\zeta(m)$$

$$= \int_{\mathbb{N}} 2^{-m}\, d\zeta(m) = \sum_{m=1}^{\infty} 2^{-m}\,\zeta(\{m\}) = 1.$$

$$\int_{\mathbb{N}}\left[\int_{\mathbb{N}} f(n,m)\, d\zeta(m)\right] d\zeta(n) = \int_{\mathbb{N}} (2-2^{-n})\,\zeta(\{n\}) + (-2+2^{-n})\,\zeta(\{n-1\})\; d\zeta(n) = 0$$

$$\int_{\mathbb{N}\times\mathbb{N}} f\, d\zeta\otimes\zeta \not\exists, \text{ da } \int f^+\, d\zeta\otimes\zeta \wedge \int f^-\, d\zeta\otimes\zeta \geq \int_{\substack{\{(n,m)\mid n=m\} \\ \text{bzw. } \{(n,m)\mid n=m+1\}}} 1\, d\zeta\otimes\zeta = \infty$$

4) $(\Omega_i, \mathcal{F}_i, \mu_i)$ $i=1,2$ σ-endl. Maßräume

lt. Radon-Nikodym $\exists\, f_1 := \frac{d\nu_1}{d\mu_1},\ f_2 := \frac{d\nu_2}{d\mu_2} \in \mathcal{M}^+_\mu$

$$\nu_1\otimes\nu_2(A) = \int_{\Omega_1}\left[\int_{\Omega_2} \mathbb{1}_A(\omega_1,\omega_2)\, \underbrace{d\nu_2(\omega_2)}_{f_2(\omega_2)\,d\mu_2(\omega_2)}\right] \underbrace{d\nu_1(\omega_1)}_{f_1(\omega_1)\,d\mu_1(\omega_1)}$$

$$= \int_{\Omega_1\times\Omega_2} \mathbb{1}_A(\omega_1,\omega_2)\, f_1(\omega_1)\, f_2(\omega_2)\, d\mu_1\otimes\mu_2(\omega_1,\omega_2)$$

andererseits gilt aber:

$$\nu_1\otimes\nu_2(A) = \int_{\Omega_1\times\Omega_2} \mathbb{1}_A(\omega_1,\omega_2)\, d\nu_1\otimes\nu_2(\omega_1,\omega_2)$$

$$\Rightarrow \frac{d\nu_1}{d\mu_1}\cdot\frac{d\nu_2}{d\mu_2} = f_1\cdot f_2 = \frac{d\nu_1\otimes\nu_2}{d\mu_1\otimes\mu_2}$$

Damit ist auch der 1. Teil des Bsp. gezeigt.

5) $(\Omega, \mathcal{F}, \mu)$ σ-endl. $\quad f \in \mathcal{M}^+(\Omega, \mathcal{F})$

ZZ: $\int f \, d\mu = \int_{[0,\infty)} \mu([f \geq t]) \, \lambda(dt)$

$$\int_{[0,\infty)} \mu([f \geq t]) \, \lambda(dt) = \int_{[0,\infty)} \mu f^{-1}([t,\infty)) \, \lambda(dt)$$

$$= \int_{[0,\infty)} \Big[\int_{[0,\infty)} \underbrace{\mathbb{1}_{[t,\infty)}(x)}\, d\mu f^{-1}(x) \Big] \, \lambda(dt)$$

$$= \int_{[0,\infty)} \Big[\int_{[0,\infty)} \underbrace{\mathbb{1}_{[0,x]}(t)} \, \lambda(dt) \Big] \, \mu f^{-1}(dx)$$

$$= \int_{[0,\infty)} \underbrace{\lambda([0,x])}_{x} \, \mu f^{-1}(dx) = \int_{f^{-1}([0,\infty))} f(x) \, \mu(dx) = \int_{\Omega} f \, d\mu.$$

6) $F: \mathbb{R} \to [0,1]$ VF

ZZ: $\int (F(x+c) - F(x)) \, \lambda(dx) = c \qquad \forall c \in \mathbb{R}$

Sei μ die durch F induzierte L-S-Maßfkt, dh $\mu((a,b]) = F(b) - F(a)$.

$$\Rightarrow \int \underbrace{F(x+c) - F(x)}_{\mu((x,x+c])} \, \lambda(dx) = \int \Big[\int \mathbb{1}_{(x,x+c]}(t) \, \mu(dt) \Big] \, \lambda(dx)$$

$$= \int \Big[\int \mathbb{1}_{[t-c,t)}(x) \, \lambda(dx) \Big] \, \mu(dt)$$

$$= \int \lambda([t-c,t)) \, \mu(dt) = c \cdot \mu(\mathbb{R}) = c \qquad \forall c > 0$$

$c = 0$: trivial

$c < 0$: analog zu oben

7) $f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}}$ ... Dichte von $P(X,Y)^{-1}$

$$PX^{-1}(A) = P(X,Y)^{-1}(A \times \mathbb{R}) = \int_A \Big[\int_{\mathbb{R}} f(x,y) \, d\lambda(y) \Big] \, d\lambda(x)$$

$$= \int_A \frac{1}{\sqrt{2\pi}} \Big[\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-\frac{(y-\rho x)^2}{2(1-\rho^2)}} \, d\lambda(y) \Big] e^{-\frac{x^2 - \rho^2 x^2}{2(1-\rho^2)}} \, d\lambda(x)$$

$$= \int_A \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, d\lambda(x). \qquad \Rightarrow X \sim N(0,1)$$