17.6.09

Lin. Algebra UE

XIII) 13.2: 1, 4, 5
13.3: 1, 6, 7
13.4: 1αβ, 4

13.2.1)

$$g = a_1 \vee a_2 = a_1 + [a_2 - a_1] = \begin{pmatrix}3\\4\\2\\3\end{pmatrix} + \left[\begin{pmatrix}-4\\-4\\2\\2\end{pmatrix}\right] =: s_1 + U_1$$

$$h = b_1 \vee b_2 = b_2 + [b_1 - b_2] = \begin{pmatrix}-4\\1\\1\\1\end{pmatrix} + \left[\begin{pmatrix}6\\0\\-4\\2\end{pmatrix}\right] =: s_2 + U_2$$

$$W := [s_2 - s_1] \oplus (U_1 + U_2) = \left[\begin{pmatrix}-7\\-3\\-1\\-2\end{pmatrix}, \begin{pmatrix}-2\\-2\\1\\1\end{pmatrix}, \begin{pmatrix}3\\0\\-2\\1\end{pmatrix}\right] = \ldots = \left[\begin{pmatrix}1\\0\\0\\\frac{19}{31}\end{pmatrix}, \begin{pmatrix}0\\1\\0\\-\frac{28}{31}\end{pmatrix}, \begin{pmatrix}0\\0\\1\\\frac{13}{31}\end{pmatrix}\right]$$

$$\left.\begin{array}{l} \Rightarrow W: \quad 19x_1 - 28x_2 + 13x_3 - 31x_4 = 0 \\ (U_1+U_2)^\perp: \begin{cases} -2x_1 - 2x_2 + x_3 + x_4 = 0 \\ 3x_1 - 2x_3 + x_4 = 0 \end{cases} \end{array}\right\} \Rightarrow \ell = \left[\begin{pmatrix}2\\1\\4\\2\end{pmatrix}\right] \in (U_1+U_2)^\perp \cap W$$

also $n = \frac{1}{5}\begin{pmatrix}2\\1\\4\\2\end{pmatrix}$

Proj. von $s_2 - s_1$ auf $[n]$:

$$((s_2 - s_1)\cdot n)\cdot n = \underbrace{-\frac{25}{5}}_{=:d} \cdot n = \begin{pmatrix}-2\\-1\\-4\\-2\end{pmatrix}$$

$$s_2 - s_1 = dn + \underset{\in U_1}{a_1} - \underset{\in U_2}{a_2} \Leftrightarrow \begin{pmatrix}-5\\-2\\3\\0\end{pmatrix} = x_1\begin{pmatrix}-2\\-2\\1\\1\end{pmatrix} - x_2\begin{pmatrix}3\\0\\-2\\1\end{pmatrix}$$

$$\Rightarrow x_1 = 1, \quad x_2 = 1$$

$$p_1 := s_1 + a_1 = \begin{pmatrix}1\\2\\3\\4\end{pmatrix}$$

$$p_2 := s_2 + a_2 = \begin{pmatrix}-1\\1\\-1\\2\end{pmatrix}$$

$A = p_1 \vee p_2$

Schnittpunkte: $g \cap A = p_1$, $h \cap A = p_2$

$\mathrm{dist}(g, h) = \mathrm{dist}(p_1, p_2) = |d| = 5$

13.2.5) ZZ: $(\mathcal{A}, dist)$ ist metr. Raum.

$dist(p,q) = \|p-q\|$

1. $dist(p,q) = \|p-q\| = \sqrt{(p-q)\cdot(p-q)} = \sqrt{(-1)^2(q-p)\cdot(q-p)} = \|q-p\| = dist(q,p)$

2. $dist(p,q) = \|p-q\| = 0 \Leftrightarrow p-q = o \Leftrightarrow p = q$

3. $dist(p,q) + dist(q,r) = \|p-q\| + \|q-r\| \underset{(11.5)}{\geq} \|p-q+q-r\| = dist(p,r)$

13.3.7) $\alpha: \mathbb{R}^{2\times 1} \longrightarrow \mathbb{R}^{2\times 1}$

$a \mapsto \underbrace{\tau_s}_{\text{Translation}} \circ \underbrace{\sigma}_{\text{Spiegelung an } g}(a) = \sigma(a) + s$

$g := p + [s] = \begin{pmatrix}1\\0\end{pmatrix} + \left[\begin{pmatrix}1\\2\end{pmatrix}\right]$ $\qquad s_\perp := \begin{pmatrix}2\\-1\end{pmatrix} \in s^\perp$

$\sigma: a \mapsto a - 2\,\frac{s_\perp \cdot (a-p)}{s_\perp \cdot s_\perp}\, s_\perp$

a)
$$\sigma\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1\\x_2\end{pmatrix} - 2\,\frac{\begin{pmatrix}2\\-1\end{pmatrix}\cdot\begin{pmatrix}x_1-1\\x_2\end{pmatrix}}{5}\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}x_1\\x_2\end{pmatrix} - \frac{2}{5}\begin{pmatrix}4x_1-4-2x_2\\-2x_1+2+x_2\end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix}-3x_1+4x_2+8\\4x_1+3x_2-4\end{pmatrix} = \frac{1}{5}\begin{pmatrix}-3 & 4\\4 & 3\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} + \frac{1}{5}\begin{pmatrix}8\\-4\end{pmatrix}$$

$$\Rightarrow \alpha: \begin{pmatrix}x_1\\x_2\end{pmatrix} \mapsto \underbrace{\frac{1}{5}\begin{pmatrix}-3 & 4\\4 & 3\end{pmatrix}}_{=: A \in O_2^-}\begin{pmatrix}x_1\\x_2\end{pmatrix} + \frac{1}{5}\begin{pmatrix}13\\6\end{pmatrix}$$

b)
$$g: \begin{pmatrix}x_0\\x_1\\x_2\end{pmatrix} \mapsto \underbrace{\frac{1}{5}\left(\begin{array}{c|cc}5 & 0 & 0\\\hline 13 & -3 & 4\\6 & 4 & 3\end{array}\right)}_{=: B \in GL_3(\mathbb{R})}\begin{pmatrix}x_0\\x_1\\x_2\end{pmatrix}$$

c) $(f_1, f_2)^T$ Fixpunkt von $\alpha \Leftrightarrow (1, f_1, f_2)^T$ EV von g zum EW 1

$$\begin{pmatrix}0 & 0 & 0\\13 & -8 & 4\\6 & 4 & -2\end{pmatrix} \to \begin{pmatrix}1 & -16 & 8\\3 & 2 & -1\\0 & 0 & 0\end{pmatrix} \to \begin{pmatrix}1 & -16 & 8\\0 & 2 & -1\\0 & 0 & 0\end{pmatrix} \to \begin{pmatrix}1 & 0 & 0\\0 & 1 & -\frac{1}{2}\\0 & 0 & 0\end{pmatrix}$$

$\Rightarrow$ EV: $\begin{pmatrix}0\\1\\2\end{pmatrix} \notin \left[\begin{pmatrix}1\\f_1\\f_2\end{pmatrix}\right] \Rightarrow \alpha$ ist fixpunktfrei

d)
$$B^2 = \frac{1}{25}\begin{pmatrix}5 & 0 & 0\\13 & -3 & 4\\6 & 4 & 3\end{pmatrix}\begin{pmatrix}5 & 0 & 0\\13 & -3 & 4\\6 & 4 & 3\end{pmatrix} = \left(\begin{array}{c|cc}1 & 0 & 0\\\hline \boxed{\begin{matrix}2\\4\end{matrix}}_{2s} & \begin{matrix}1\\0\end{matrix} & \begin{matrix}0\\1\end{matrix}\end{array}\right)$$

$$\Rightarrow \alpha^2\begin{pmatrix}x_1\\x_2\end{pmatrix} = E_2\cdot\begin{pmatrix}x_1\\x_2\end{pmatrix} + \begin{pmatrix}2\\4\end{pmatrix}$$

13.4.1) α) $\lambda : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto 9x_1^2 - 24x_1x_2 - 10x_1 + 16x_2^2 + 180x_2 + 325$

$$G_0 = \left(\begin{array}{c|cc} 325 & -5 & 90 \\ \hline \boxed{-5} & 9 & -12 \\ \boxed{90} & -12 & 16 \end{array}\right) \qquad \begin{pmatrix} -5 \\ 90 \end{pmatrix} =: g, \quad \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} =: G$$

$$\det(G - xE_2) = \begin{vmatrix} 9-x & -12 \\ -12 & 16-x \end{vmatrix} = (9-x)(16-x) - 144 = x(x-25)$$

$\Rightarrow$ EW: $\underbrace{\lambda_1 = 0}_{\Rightarrow \text{Typ B}}$, $\lambda_2 = 25$ $\Rightarrow \tilde{G} := \operatorname{diag}(25, 0)$

EV zu $\lambda_2 = 25$:

$$\begin{pmatrix} -16 & -12 \\ -12 & -9 \end{pmatrix} \to \begin{pmatrix} -4 & -3 \\ -4 & -3 \end{pmatrix} \Rightarrow \tilde{b}_1 := \frac{1}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$\tilde{b}_2$ ist so zu wählen, dass $g \cdot \tilde{b}_2 < 0 \Rightarrow \tilde{b}_2 := \frac{1}{5}\begin{pmatrix} -4 \\ -3 \end{pmatrix}$

$$\Rightarrow \tilde{G}_0 = \left(\begin{array}{c|cc} 325 & -75 & -50 \\ \hline \boxed{-75} & 25 & 0 \\ \boxed{-50} & 0 & 0 \end{array}\right) \qquad \begin{pmatrix} -75 \\ -50 \end{pmatrix} = \tilde{B}^t \cdot g, \quad \begin{pmatrix} 25 & 0 \\ 0 & 0 \end{pmatrix} = \tilde{G}$$

Translation des Ursprunges:

$$\begin{array}{c|cc} 325 & -75 & -50 \\ -75 & 25 & 0 \\ -50 & 0 & 0 \\ \hline 1 & 0 & 0 \\ \hline 0 & & \\ 0 & \multicolumn{2}{c}{\tilde{B}} \end{array} \;\cdot 3 \;\xmapsto{s} \xmapsto{z}\; \begin{array}{c|cc} 100 & 0 & -50 \\ 0 & 25 & 0 \\ -50 & 0 & 0 \\ \hline 1 & 0 & 0 \\ \hline \frac{9}{5} & & \\ -\frac{12}{5} & \multicolumn{2}{c}{\tilde{B}} \end{array} \;\xmapsto{s} \xmapsto{z}\; \begin{array}{c|cc} 0 & 0 & -50 \\ 0 & 25 & 0 \\ -50 & 0 & 0 \\ \hline 1 & 0 & 0 \\ \hline 1 & & \\ -3 & \multicolumn{2}{c}{\tilde{B}} \end{array} \quad \tilde{u} := \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$\Rightarrow$ Koo.-System: $(\tilde{u}, \tilde{u} + \tilde{B})$

$$\Phi(\lambda): \; 25\tilde{x}_1^2 - 100\tilde{x}_2 = 0 \iff \frac{1}{2}\tilde{x}_1^2 = 2\tilde{x}_2$$

affin, euklidisch: Parabel
projektiv: reg. Kegelschnitt

β) $\lambda : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto 52x_1^2 + 28x_1x_2 - 208x_1 + 73x_2^2 - 56x_2 - 512$

$$G_0 = \left(\begin{array}{c|cc} -512 & -104 & -28 \\ \hline \boxed{-104} & 52 & 14 \\ \boxed{-28} & 14 & 73 \end{array}\right) \qquad \begin{pmatrix} -104 \\ -28 \end{pmatrix} =: B \cdot g, \quad \begin{pmatrix} 52 & 14 \\ 14 & 73 \end{pmatrix} =: G$$

$$\det(G - xE_2) = \begin{vmatrix} 52-x & 14 \\ 14 & 73-x \end{vmatrix} = (52-x)(73-x) - 196 = (x-45)(x-80)$$

$\Rightarrow$ EW: $\lambda_1 = 45$, $\lambda_2 = 80$ $\Rightarrow$ Typ A

EV zu $\lambda_1 = 45$:

$$\begin{pmatrix} 7 & 14 \\ 14 & 28 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \qquad \Rightarrow \tilde{e}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \Rightarrow \tilde{e}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Mittelpunkt m: $(m_1, m_2) \cdot G = -g^T \Leftrightarrow (m_1, m_2) = -g^T \cdot G^{-1}$

$$= (104, 28) \frac{1}{3600} \begin{pmatrix} 73 & -14 \\ -14 & 52 \end{pmatrix} = \underbrace{(2,0)}_{=:\tilde{u}}$$

$$\Rightarrow \lambda(\tilde{u}) = 52 \cdot 4 - 208 \cdot 2 - 512 = -720$$

$$\Phi(\lambda): \; 45 \tilde{x}_1^2 + 80 \tilde{x}_2^2 - 720 = 0 \quad \Leftrightarrow \quad \frac{1}{16} \tilde{x}_1^2 + \frac{1}{9} \tilde{x}_2^2 = 1$$

affin, eukl.: Ellipse
projektiv: reg. Kegelschnitt

13.4.4) Sei k eine bel. Parabel in der eukl. Ebene, dann $\exists$ ein kart. Koo-System $(u, u+B)$ derart, dass die Gl. von k eukl. Normalform besitzt:

$g_1 x_1^2 = 2x_2$ mit $g_1 > 0$

Sei $p := (p_1, p_2)^T \in k$. $\Rightarrow A_p := p^\perp$: $g_1 x_1 p_1 - x_2 = p_2$

$$\Rightarrow A_p = \begin{pmatrix} 0 \\ -p_2 \end{pmatrix} + \underbrace{\left[\left(\begin{matrix} 1 \\ g_1 p_1 \end{matrix}\right)\right]}_{=: a^\perp} \quad \Rightarrow [a] = \begin{pmatrix} g_1 p_1 \\ -1 \end{pmatrix}$$

<u>Spiegelung an A_p:</u>

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - 2 \frac{a \cdot \begin{pmatrix} x_1 \\ x_2 + p_2 \end{pmatrix}}{a \cdot a} a = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \frac{2}{(g_1 p_1)^2 + 1} \left(g_1 p_1 x_1 - (x_2 + p_2) \right) \begin{pmatrix} g_1 p_1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \frac{2}{g_1^2 p_1^2 + 1} \begin{pmatrix} g_1^2 p_1^2 x_1 - x_2 g_1 p_1 - g_1 p_1 p_2 \\ -g_1 p_1 x_1 + x_2 + p_2 \end{pmatrix}$$

$$= \frac{1}{g_1^2 p_1^2 + 1} \begin{pmatrix} -g_1^2 p_1^2 x_1 + 2 x_2 g_1 p_1 + 2 g_1 p_1 p_2 + x_1 \\ 2 g_1 p_1 x_1 + x_2 g_1^2 p_1^2 - 2 p_2 - x_2 \end{pmatrix}$$

$p \in k \Rightarrow p_2 = \frac{1}{2} g_1 p_1^2$

Geraden parallel zur x_2-Achse, schneiden k in p. $\Rightarrow x_1 = p_1$

$A_p \cap$ x_2-Achse ~~k~~:

$$\Rightarrow \cancel{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} \Leftrightarrow \frac{1}{g_1^2 p_1^2 + 1} \begin{pmatrix} 2 g_1 p_1 x_2 + p_1 \\ (g_1^2 p_1^2 - 1) x_2 + g_1 p_1^2 \end{pmatrix} = \begin{pmatrix} 0 \\ f_2 \end{pmatrix} =: f$$

$$2 g_1 p_1 x_2 + p_1 = 0 \Leftrightarrow x_2 = -\frac{1}{2 g_1}$$

$$(g_1^2 p_1^2 - 1) x_2 + g_1 p_1^2 = f_2 (g_1^2 p_1^2 + 1) - g_1 p_1^2$$

$$\Leftrightarrow \quad 1 - g_1^2 p_1^2 = f_2 \cdot 2 g_1 (g_1^2 p_1^2 + 1) - 2 g_1^2 p_1^2$$

$$\Leftrightarrow \quad \overset{1}{\cancel{1 + g_1^2 p_1^2}} = 2 g_1 \cdot f_2 \cancel{(g_1^2 p_1^2 + 1)} \Leftrightarrow \underline{f_2 = \frac{1}{2 g_1}}$$ also von x unabh.! u. p