18.3.09

Lin. Algebra

II, 8.5: 3, 4, 5
8.6: 2, 8
8.7: 1a, 2, 6

8.5.3) $A := \begin{pmatrix} 0 & \frac{2}{3} \\ 1 & \frac{1}{3} \end{pmatrix}$

a) $\chi_A(x) = \det(A - xE_n) = \begin{vmatrix} -x & \frac{2}{3} \\ 1 & \frac{1}{3}-x \end{vmatrix} = -x\left(\frac{1}{3}-x\right) - \frac{2}{3} = x^2 - \frac{x}{3} - \frac{2}{3}$

$\chi_A(x) = 0 \Leftrightarrow x^2 - \frac{1}{3}x - \frac{2}{3} = 0$

$\Leftrightarrow x_{1,2} = \frac{1}{6} \pm \sqrt{\frac{1}{36} + \frac{2}{3}}$

$= \frac{1}{6} \pm \frac{5}{6}$

$\Rightarrow \lambda_1 = 1 \ ; \ \lambda_2 = -\frac{2}{3}$

$ER(1) = \ker(A - E_n)$

x_1	x_2	
-1	$\frac{2}{3}$	0
1	$-\frac{2}{3}$	

$\Rightarrow$ Basis: $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\Rightarrow$ gesuchter EV: $\begin{pmatrix} \frac{2}{5} \\ \frac{3}{5} \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$

$ER(-\frac{2}{3}) = \ker(A + \frac{2}{3}E_n)$

x_1	x_2	
$\frac{2}{3}$	$\frac{2}{3}$	0
1	1	

$\Rightarrow$ Basis: $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 \\ 0 & -\frac{2}{3} \end{pmatrix} = P^{-1} \begin{pmatrix} 0 & \frac{2}{3} \\ 1 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$

Ü. 8.5.2: $B^k = P^{-1} A^k P \Rightarrow \lim\limits_{k\to\infty} A^k = P \cdot \lim\limits_{k\to\infty} B^k \cdot P^{-1}$

$\lim\limits_{k\to\infty} B^k = \lim\limits_{k\to\infty} \begin{pmatrix} 1 & 0 \\ 0 & -\frac{2}{3} \end{pmatrix}^k = \begin{pmatrix} \lim\limits_{k\to\infty} 1^k & 0 \\ 0 & \lim\limits_{k\to\infty} \left(-\frac{2}{3}\right)^k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$\lim\limits_{k\to\infty} A^k = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} P^{-1} = \begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix} \cancel{P^{-1}} \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{3}{5} \end{pmatrix} = \begin{pmatrix} g_1 & g_1 \\ g_2 & g_2 \end{pmatrix}$

8.6.2) $A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

Cayley-Hamilton: $\chi_f(A) = 0$

$$\chi_f(X) = \det(A - XE_n) = \begin{vmatrix} 1-X & 0 & -1 \\ 1 & 2-X & 0 \\ 0 & -1 & 1-X \end{vmatrix}$$

$$= (1-X)^2(2-X) + 1 = (1-2X+X^2)(2-X) + 1$$

$$= 2 - 4X + 2X^2 - X + 2X^3 - X^3 + 1 = -X^3 + 4X^2 - 5X + 3$$

$$\chi_f(A) = -A^3 + 4A^2 - 5A + 3 = 0$$
$$\Leftrightarrow A^3 = 4A^2 - 5A + 3A^0$$

$$A^4 = A \cdot A^3 = A\left(4A^2 - 5A + 3A^0\right) = 4A^3 - 5A^2 + 3A$$
$$= 4\left(4A^2 - 5A + 3A^0\right) - 5A^2 + 3A = 11A^2 - 17A + 12A^0$$

8.7.2) $$\chi_A(X) = \det(A - XE_n) = \begin{vmatrix} -X & 1 & 0 & 0 \\ -4 & 4-X & 0 & 0 \\ -6 & 4 & -X & 1 \\ -8 & 6 & -4 & 4-X \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 4-X & (4-X)X-4 & 0 & 0 \\ 4 & 4X-6 & 1 & 0 \\ 6 & 6X-8 & 4-X & (4-X)X-4 \end{vmatrix}$$

$$= (-X^2 + 4X - 4)^2 = (X^2 - 4X + 4)^2 = (X-2)^4$$

$\Rightarrow \lambda = 2$, alg. VF $= 4$

Stufe 1: $\ker(A - 2E_n) = \begin{pmatrix} -2 & 1 & 0 & 0 \\ -4 & 2 & 0 & 0 \\ -6 & 4 & -2 & 1 \\ -8 & 6 & -4 & 2 \end{pmatrix} \Rightarrow \operatorname{rg} \ker(A - 2E_n) = 2 \Rightarrow$ Lösungsraum $4-2=2$ dim. (Eigenraum)

also $\exists$ 2 l.u. Eigenvektoren.

Stufe 2: $\ker(A - 2E_n)^2 = (0) \Rightarrow$ Lösungsraum 4 dim.

also 2 ~~weitere~~ Hauptvektoren mit Stufe 2.

$\Rightarrow$ insgesamt: $2\ J_2(2)$

$$A' = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$\chi_{A'}(X) = (X-2)^4$

$\mu_{A'}(X) = (X-2)^2$

$$\chi_B(X) = \det(B - XE_n) = \begin{vmatrix} -1-X & 1 & 0 & \boxed{0} \\ -9 & 5-X & 0 & 0 \\ -6 & 2 & 2-X & 0 \\ -3 & 1 & 0 & 2-X \end{vmatrix} = (2-X) \begin{vmatrix} -1-X & 1 & 0 \\ -9 & 5-X & 0 \\ -6 & 2 & 2-X \end{vmatrix}$$

$$= (2-X)\Big((-1-X)(5-X)(2-X) + 9(2-X)\Big)$$

$$= (2-X)^2 (X^2 - 4X + 4) = (2-X)^4$$

$\Rightarrow A = 2$; algebr. VF $= 4$

Stufe 1: $\ker(B - 2E_n) = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -9 & 3 & 0 & 0 \\ -6 & 2 & 0 & 0 \\ -3 & 1 & 0 & 0 \end{pmatrix} \Rightarrow$ rg $\ker(B-2E_n) = 1 \Rightarrow$ ER 3-dim.

Stufe 2: aus oben folgt: $\exists$ 1 HV mit Stufe 2

$\Rightarrow$ insgesamt: $1\, J_2(2)$, $2\, J_1(2)$

$$B' = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \boxed{2} & 0 \\ 0 & 0 & 0 & \boxed{2} \end{pmatrix} \qquad \chi_{B'}(X) = (X-2)^4 \qquad \mu_{B'}(X) = (X-2)^2$$

$\Rightarrow \chi_{A'} = \chi_{B'}$, $\mu_{A'} = \mu_{B'}$ aber $A' \not\approx B'$ lt. Satz 8.7.11

8.7.1)α)

a)
$$\chi_f(X) = \det\Big(\overset{f - X\,id_V}{\cancel{\langle B^*, f(B)\rangle - XE_n}}\Big) = \begin{vmatrix} 2-X & -1 & 0 & 1 \\ 0 & 2-X & 0 & 0 \\ 0 & -1 & 2-X & 1 \\ -1 & 0 & 1 & 2-X \end{vmatrix}$$

$$= (2-X) \begin{vmatrix} 2-X & 0 & 1 \\ 0 & 2-X & 1 \\ -1 & 1 & 2-X \end{vmatrix} = (2-X)\Big((2-X)^3 + (2-X) - (2-X)\Big) = (2-X)^4$$

b) $A = 2$, algb. VF $= 4$

$ER(2) = \ker(f - 2\, id_V)$

x_1	x_2	x_3	x_4	
0	-1	0	1	
0	0	0	0	0
0	-1	0	1	
-1	0	1	0	

$\Rightarrow$ Basis der EV: $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\langle B^*, a_{11} \rangle$ $\quad \langle B^*, a_{21} \rangle$

$\ker(f-2\,id_V)^2 \Leftrightarrow a_{11} \leftarrow\!\shortmid x_1 \quad a_{21} \leftarrow\!\shortmid x_2$

$a_{12} \leftarrow \ker(f-2\,id_V)^3 \Leftrightarrow a_{11} \leftarrow\!\shortmid x_3$

	x_1	x_2	x_3	x_4	a_{11}	a_{21}	a_{12}
$\cdot(-1)$	0	-1	0	1	1	0	0
	0	0	0	0	0	1	0
$\cdot(-1)$	0	-1	0	1	1	0	0
$\cdot(-1)$	-1	0	1	0	0	1	1
	1	0	-1	0	0	-1	1
	0	1	0	-1	-1	0	0
	0	0	0	0	0	1	0
	0	0	0	0	0	0	0

$$\Rightarrow \langle B^*, a_{12}\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \quad \Rightarrow \langle B^*, a_{13}\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

Schema: $c_1 := a_{11} \leftarrow\!\shortmid c_2 := a_{12} \leftarrow\!\shortmid c_3 := a_{13}$

$c_4 := a_{21}$

$$\Rightarrow B = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

c) Basis C von $B = \langle C^*, f(C)\rangle$:

$$C = \{(c_1, c_2, c_3, c_4)\} = \left\{ \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix} \right\}$$

d) $\mu_f(X) = (X-2)^3$

8.7.6) $\frac{d}{dX}: \mathbb{R}[X] \to \mathbb{R}[X]: a_{n-1}X^{n-1} + a_{n-2}X^{n-2} + \dots + a_1 X + a_0 \mapsto$
$(n-1)a_{n-1}X^{n-2} + \dots + 2a_2 X + a_1$

a) nur für konstante $P[X]$ $\exists$ ein EW, $\lambda = 0$.

b) $U_n := [\{1, X, X^2, \dots, X^n\}]$

$$\left\langle E_n^*, \frac{d}{dX}(E_n^*)\right\rangle = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 0 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & n-1 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \Rightarrow \text{Basis } ER(0): \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} =: b_0$$

x_0	x_1	x_2	x_3	x_4	$\dots$	x_{n-1}	b_0	b_1	b_2	b_3	$\dots$	b_{n-1}
0	1	0	0	0	$\dots$	0	1	0	0	0	$\dots$	0
0	0	2	0	0	$\dots$	0	0	1	0	0	$\dots$	0
0	0	0	3	0	$\dots$	0	0	0	$\frac{1}{2}$	0	$\dots$	0
0	0	0	0	4	$\dots$	0	0	0	0	$\frac{1}{6}$	$\dots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
0	0	0	0	0	$\dots$	$n-1$	0	0	0	0	$\dots$	0
0	0	0	0	0	$\dots$	0	0	0	0	0	$\dots$	$\frac{1}{(n-1)!}$

$\Rightarrow \langle E_n^*, B_n \rangle = (b_0, b_1, \dots, b_{n-1})$

$J_n(0) = \langle B_n^*, E_n \rangle \left\langle E_n^*, \frac{d}{dX}(E_n)\right\rangle \langle E_n^*, B_n \rangle$

8.5.4) a) A, B haben Basis aus EV $\Rightarrow$ A, B diagonalisierbar

Sei P die Matrix der Basisvektoren der EV $\Rightarrow \exists C, D \in K^{n\times n}$ derart, dass

$C = P^{-1}AP$
$D = P^{-1}BP$ C, D sind die zu A, B ähnlichen Diagonalmatrizen

ZZ: $AB = BA$

$A \cdot B = P^{-1}C\underbrace{P \cdot P^{-1}}_{E_n}DP = P^{-1}CDP \overset{\text{Diag. kommut.}}{=} P^{-1}DCP = P^{-1}D\underbrace{PP^{-1}}_{E_n}CP = B \cdot A$

b) Unterraumkriterium:

$U \neq \emptyset$, da $A \in U$

U abgeschl.? ZZ: $(B + yC) \in U \quad \forall B, C \in U$

$A(B+yC) = AB + AyC = BA + yCA = (B+yC)A$ ✓

c) A ist nicht skalar $\Rightarrow$ mind. komm. zu E_n und A

A ist skalar $\Rightarrow$ ~~mind.~~ komm. zu ~~A u.~~ jeder bel. Matrix, also $\dim U \geq 4$

d) A diagonalisierbar $\Rightarrow \exists\, C \in K^{n\times n}$ mit $\overset{\text{Diag.}}{C} = P^{-1}AP \Leftrightarrow A = PCP^{-1}$

setze $B = PDP^{-1}$ wobei $D = \operatorname{diag}(d_1, \dots, d_n)$

$\overset{a)}{\Rightarrow} AB = BA$

$$\text{Basis v. } U = \left(P\begin{pmatrix} d_1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}P^{-1}\right), \dots, \left(P\begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}P^{-1}\right)$$

8.5.5)

$$\chi_A(X) = \det(A - XE_n) = \begin{vmatrix} -X & 0 & 4 \\ 2 & 2-X & -4 \\ 1 & 0 & -X \end{vmatrix} = X^2(2-X) - 4(2-X)$$

$$= (2-X)(X^2-4)$$

$$= (-1)(X-2)^2(X+2)$$

$ER(2) = \ker(A - 2E_n)$

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \hline -2 & 0 & 4 & \\ 2 & 0 & -4 & 0 \\ 1 & 0 & -2 & \end{array} \Rightarrow \text{Basis: } \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$ER(-2) = \ker(A + 2E_n)$

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \hline 2 & 0 & 4 & \\ 2 & 4 & -4 & 0 \\ 1 & 0 & 2 & \\ \hline 1 & 0 & 2 & \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & \end{array} \Rightarrow \text{Basis: } \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$$

Vektoren l.u. $\Rightarrow \exists$ Basis aus EV $\Rightarrow$ A diagonalisierbar

8.5.4 d)

$$\Rightarrow \text{Basis v. } U\text{: } P \cdot \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \cdot P^{-1}$$

$$\text{also } U = \left\{ P \cdot \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \cdot P^{-1} \;\middle|\; \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \right\}$$