11.1.10

Differentialgleichungen UE

IX, 250, 251, 255, 259, 263, 270

250) $\ddot{x} + (3t-1)\dot{x} + 4x = \cos t$ $\qquad x(0)=1,\ \dot{x}(0)=2$

$\Leftrightarrow \ddot{x} - \dot{x} + 3t\dot{x} + 4x = \cos t$

$\Rightarrow \mathcal{L}(\ddot{x})(u) + \mathcal{L}(\dot{x})(u) + 3\mathcal{L}(t\dot{x})(u) + \mathcal{L}(tx)(u) = \mathcal{L}(\cos t)(u) = \frac{u}{u^2+1}$

$\mathcal{L}(\dot{x})(u) = u\mathcal{L}(x)(u) - x(0) = u\mathcal{L}(x)(u) - 1$

$\mathcal{L}(\ddot{x})(u) = u\mathcal{L}(\dot{x})(u) - \dot{x}(0) = u^2\mathcal{L}(x)(u) - \cancel{1}u - 2$

$\mathcal{L}(tx)(u) = -\frac{d}{du}\mathcal{L}(x)(u)$

$\mathcal{L}(t\dot{x})(u) = -\frac{d}{du}\mathcal{L}(\dot{x})(u) = -\mathcal{L}(x)(u) - u\frac{d}{du}\mathcal{L}(x)(u)$

$\rightsquigarrow u^2\mathcal{L}(x)(u) - u - 2 - u\mathcal{L}(x)(u) + 1 - 3\mathcal{L}(x)(u) - 3u\frac{d}{du}\mathcal{L}(x)(u) - \frac{d}{du}\mathcal{L}(x)(u) = \frac{u}{u^2+1}$

$\Leftrightarrow -(3u+1)\dot{\mathcal{L}}(x) + (u^2-u-3)\mathcal{L}(x) = \frac{u}{u^2+1} + u + 1$

251) $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \underbrace{\begin{pmatrix}2 & -6\\ 3 & -4\end{pmatrix}}_{A}\begin{pmatrix}x\\ y\end{pmatrix} + \underbrace{\begin{pmatrix}14\\ 13\end{pmatrix}\sin t}_{b(t)}$

$x(0) = x(\pi)$
$y(0) = 1$ d.h. $\underbrace{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}_{C}\begin{pmatrix}x\\ y\end{pmatrix}(0) + \underbrace{\begin{pmatrix}-1 & 0\\ 0 & 0\end{pmatrix}}_{D}\begin{pmatrix}x\\ y\end{pmatrix}(\pi) = \underbrace{\begin{pmatrix}0\\ 1\end{pmatrix}}_{c}$

•) <u>Lösungsbasis d. hom. DGL</u>

$\chi_A(\lambda) = \lambda^2 + 2\lambda + 10 = 0 \Leftrightarrow \lambda = -1 \pm \sqrt{1-10} = -1 \pm 3i$

$\ker(A - (-1\pm 3i)E) = \ker\begin{pmatrix}3-3i & -6\\ 3 & -3-3i\end{pmatrix} = \ker\begin{pmatrix}1-i & -2\\ 1 & -1-i\end{pmatrix} = \left[\begin{pmatrix}1+i\\ 1\end{pmatrix}\right]$

$\Rightarrow W(t) = Te^{t\,J} = \left(\begin{pmatrix}1+i\\ 1\end{pmatrix}e^{(-1+3i)t}, \begin{pmatrix}1-i\\ 1\end{pmatrix}e^{(-1-3i)t}\right)$

$\Rightarrow$ reelle LB: $\begin{pmatrix}1\\ 1\end{pmatrix}e^{-t}\cos 3t - \begin{pmatrix}1\\ 0\end{pmatrix}e^{-t}\sin 3t,\ \begin{pmatrix}1\\ 1\end{pmatrix}e^{-t}\sin 3t + \begin{pmatrix}1\\ 0\end{pmatrix}e^{-t}\cos 3t$

·) *part. Lösung*

$i \neq EW \Rightarrow$ Ansatz $\quad x = \sigma_1 \sin t + \tau_1 \cos t$

$\qquad\qquad\qquad\qquad\quad y = \sigma_2 \sin t + \tau_2 \cos t$

$$\rightsquigarrow \begin{cases} \sigma_1 \cos t - \tau_1 \sin t = 2(\sigma_1 \sin t + \tau_1 \cos t) - 6(\sigma_2 \sin t + \tau_2 \cos t) + 14 \sin t \\ \sigma_2 \cos t - \tau_2 \sin t = 3(\text{———}\,"\,\text{———}) - 4(\text{———}\,"\,\text{———}) + 13 \sin t \end{cases}$$

$$\Leftrightarrow \begin{cases} 0 = (\tau_1 + 2\sigma_1 - 6\sigma_2 + 14)\sin t + (-\sigma_1 + 2\tau_1 - 6\tau_2)\cos t \\ 0 = (\tau_2 + 3\sigma_1 - 4\sigma_2 + 13)\sin t + (-\sigma_2 + 3\tau_1 - 4\tau_2)\cos t \end{cases}$$

σ_1	σ_2	τ_1	τ_2	1
-1	0	2	-6	0
0	-1	3	-4	0
2	-6	1	0	-14
3	-4	0	1	-13
1	0	-2	6	0
0	1	-3	4	0
0	-6	5	-12	-14
0	-4	6	-17	-13
1	0	-2	6	0
0	1	-3	4	0
0	0	-13	12	-14
0	0	-6	-1	-13
1	0	-38	0	-78
0	1	-27	0	-52
0	0	-85	0	-170
0	0	6	1	13

$\cdot 2 \quad \cdot 3 \quad \cdot 6 \quad \cdot 4 \quad \cdot 6 \quad \cdot 4 \quad \cdot 12$

$\Rightarrow \tau_1 = 2, \quad \tau_2 = 13 - 6\tau_1 = 1$

$\sigma_1 = -78 + 38\tau_1 = -2$

$\sigma_2 = -52 + 27\tau_1 = 2$

$$\Rightarrow \varphi_p = \begin{pmatrix} -2\sin t + 2\cos t \\ 2\sin t + \cos t \end{pmatrix}$$

·) Randwertproblem

$$C \cdot \varphi(0) + D \cdot \varphi(\pi) = c \Leftrightarrow C\big(W(0)\sigma + \varphi_p(0)\big) + D\big(W(\pi)\sigma + \varphi_p(\pi)\big) = c$$

$$\Leftrightarrow \underbrace{\big(CW(0) + DW(\pi)\big)}_{R(W)}\sigma = c - \underbrace{\big(C\varphi_p(0) + D\varphi_p(\pi)\big)}_{R(\varphi_p)}$$

$$R(W) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -e^{-\pi} & -e^{-\pi} \\ -e^{-\pi} & 0 \end{pmatrix} = \begin{pmatrix} 1+e^{-\pi} & 1+e^{-\pi} \\ 1 & 0 \end{pmatrix}$$

$$R(\varphi_p) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$R(W) \text{ regulär} \Rightarrow \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = R(W)^{-1}\big(c - R(\varphi_p)\big)$$

$$= -\frac{1}{1+e^{-\pi}}\begin{pmatrix} 0 & -1-e^{-\pi} \\ -1 & 1+e^{-\pi} \end{pmatrix}\begin{pmatrix} -4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{4}{1+e^{-\pi}} \end{pmatrix}.$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{-t}\cos 3t - e^{-t}\sin 3t & e^{-t}\sin 3t + e^{-t}\cos 3t \\ e^{-t}\cos 3t & e^{-t}\sin 3t \end{pmatrix}\begin{pmatrix} 0 \\ -\frac{4}{1+e^{-\pi}} \end{pmatrix} + \begin{pmatrix} -2\sin t + 2\cos t \\ 2\sin t + \cos t \end{pmatrix}$$

259) $x^{(3)} + \ddot{x} + \dot{x} + x = e^{-t} - 1$

$$\begin{matrix} x(0)=1 \\ x(\pi) = e^{-\pi}+1 \\ \dot{x}(0) = \dot{x}(\pi) \end{matrix}, \quad \text{d.h.} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \\ \ddot{x} \end{pmatrix}(0) + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \\ \ddot{x} \end{pmatrix}(\pi) = \begin{pmatrix} 1 \\ e^{-\pi}+1 \\ 0 \end{pmatrix}$$

Wissen aus Bsp. 243:

$$W(t) = \begin{pmatrix} e^{-t} & \cos t & \sin t \\ -e^{-t} & -\sin t & \cos t \\ e^{-t} & -\cos t & -\sin t \end{pmatrix}$$

$$R(W) = C \cdot W(0) + D \cdot W(\pi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} e^{-\pi} & -1 & 0 \\ -e^{-\pi} & 0 & -1 \\ e^{-\pi} & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ e^{-\pi} & -1 & 0 \\ -1+e^{-\pi} & 0 & 2 \end{pmatrix} \text{ ist regulär } \Rightarrow \exists! \text{ Lösung des RWP.}$$

263) $$G(t,u) = W(t) \underbrace{\left(z(t,u) \cdot E - R(W)^{-1} \cdot D \cdot W(\pi) \right)}_{=: H(t,u)} \cdot W^{-1}(u)$$

$$R(W)^{-1} \underline{D\, W(\pi)} = \frac{1}{2(1+e^{-\pi})} \begin{pmatrix} 2 & 2 & 0 \\ 2e^{-\pi} & 2 & 0 \\ 1-e^{-\pi} & 1-e^{-\pi} & 1+e^{-\pi} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ e^{-\pi} & -1 & 0 \\ e^{-\pi} & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2(1+e^{-\pi})} \begin{pmatrix} 2e^{-\pi} & -2 & 0 \\ -2e^{-\pi} & 2 & 0 \\ 2e^{-\pi} & e^{-\pi}-1 & 1+e^{-\pi} \end{pmatrix}$$

$$W^{-1}(u) = \frac{1}{2} \begin{pmatrix} e^{u} & 0 & e^{u} \\ -\sin u + \cos u & -2\sin u & -\cos u - \sin u \\ \cos u + \sin u & 2\cos u & -\sin u + \cos u \end{pmatrix}$$

$$G(t,u) = \begin{cases} \overbrace{-W(t)\left(R(W)^{-1} D\, W(\pi)\right) W^{-1}(u)}^{G_1(t,u) :=} & 0 \le t < u \le \pi \\ W(t)\, W^{-1}(u) + G_1(t,u) & 0 \le u \le t \le \pi \end{cases}$$

$$\Rightarrow \Gamma_1(t,u) = -\frac{1}{4(1+e^{-\pi})} \underbrace{(e^{-t}, \cos t, \sin t)}_{\text{1. Zeile v. } W(t)} \begin{pmatrix} 2e^{-\pi} & -2 & 0 \\ -2e^{-\pi} & 2 & 0 \\ 2e^{-\pi} & e^{-\pi}-1 & 1+e^{-\pi} \end{pmatrix} \underbrace{\begin{pmatrix} e^{u} \\ -\cos u - \sin u \\ -\sin u + \cos u \end{pmatrix}}_{\text{letzte Spalte von } W^{-1}(u)}$$

$$= -\frac{1}{2(1+e^{-\pi})} \Big(e^{-t}(e^{u-\pi} + \cos u + \sin u) - \cos t\, (e^{u-\pi} + \cos u + \sin u)$$
$$+ \sin t\, (e^{u-\pi} + \cos u - e^{-\pi} \sin u) \Big) \qquad 0 \le t < u \le \pi$$

$$\Gamma_2(t,u) = \frac{1}{2}(e^{-t}, \cos t, \sin t) \begin{pmatrix} e^{u} \\ -\cos u - \sin u \\ -\sin u + \cos u \end{pmatrix} + \Gamma_1(t,u)$$

$$= -\frac{1}{2(1+e^{-\pi})} \Big(-(1+e^{-\pi}) e^{u-t} + (1+e^{-\pi}) \cos t (\cos u + \sin u) - (1+e^{-\pi}) \sin t (\cos u - \sin u)$$
$$+ e^{-t}(e^{u-\pi} + \cos u + \sin u) - \cos t\, (e^{u-\pi} + \cos u + \sin u) + \sin t\, (e^{u-\pi} + \cos u - e^{-\pi} \sin u) \Big)$$

$$= -\frac{1}{2(1+e^{-\pi})} \Big(e^{-t}(-e^{u} + \cos u + \sin u) - \cos t\, (e^{u-\pi} - e^{-\pi}(\cos u + \sin u)) + \sin t\, (e^{u-\pi} + \sin u - e^{-\pi} \cos u) \Big) \qquad 0 \le u \le t \le \pi$$

270) $\ddot{x} + 2\dot{x} - 8x + \lambda x = 0$ $x(1) = 0$, $\dot{x}(0) = 0$

Char. Polynom d. äqu. Systems 1. Ordnung:

$$\chi_A(x) = x^2 + 2x - 8 + \lambda = 0 \Leftrightarrow x = -1 \pm \sqrt{1+8-\lambda} = -1 \pm \sqrt{9-\lambda}$$

1. Fall:

$9 - \lambda > 0 \Leftrightarrow \lambda < 9$

Sei $a := \sqrt{9-\lambda} > 0 \Rightarrow \varphi(t) = c_1 e^{(-1+a)t} + c_2 e^{(-1-a)t}$

$$RB \rightsquigarrow \begin{cases} \varphi(1) = c_1 e^{-1+a} + c_2 e^{-1-a} = 0 \Leftrightarrow c_1 = -c_2 e^{-2a} \\ \dot{\varphi}(0) = (-1+a) c_1 + (-1-a) c_2 = 0 \end{cases}$$

$$\rightsquigarrow \underbrace{(1-a)}_{\leq 1} c_2 \underbrace{e^{-2a}}_{\leq 1} = \underbrace{(+1+a)}_{\geq 1} c_2 \Leftrightarrow a = 0 \text{ ↯ zu VS} \Rightarrow \text{für } \lambda < 9 \text{ existiert}$$

keine nicht triviale Lösung.

2. Fall: $\lambda = 9 \Rightarrow -1$ ist doppelter EW

$\Rightarrow \varphi(t) = c_1 e^{-t} + c_2 t e^{-t}$

$$RB \rightsquigarrow \begin{cases} \varphi(1) = c_1 e^{-1} + c_2 e^{-1} = 0 \Leftrightarrow -c_1 = c_2 \\ \dot{\varphi}(0) = -c_1 + c_2 = 0 \Leftrightarrow c_1 = c_2 \end{cases} \Rightarrow c_1 = c_2 = 0.$$

3. Fall:

$9 - \lambda < 0 \Leftrightarrow \lambda > 9$ d.h. $x = -1 \pm \sqrt{9-\lambda} = -1 \pm \underbrace{\sqrt{\lambda-9}}_{=: b > 0}\, i$

$\Rightarrow \varphi(t) = c_1 e^{-t} \cos bt + c_2 e^{-t} \sin bt$

$\dot{\varphi}(t) = -c_1 e^{-t} \cos bt - b c_1 e^{-t} \sin bt - c_2 e^{-t} \sin bt + b c_2 e^{-t} \cos bt$

$RB \rightsquigarrow \varphi(1) = c_1 e^{-1} \cos b + c_2 e^{-1} \sin b = 0$

$\dot{\varphi}(0) = -c_1 + b c_2 = 0 \Leftrightarrow c_1 = b c_2$

$$\rightsquigarrow b c_2 e^{-1} \cos b + c_2 e^{-1} \sin b = 0 \Leftrightarrow \sqrt{\lambda-9} \cos \sqrt{\lambda-9} + \sin \sqrt{\lambda-9} = 0 \quad (\lambda > 9)$$