30.11.09

Differentialgleichungen UE

VI, 156, 157, 164, 168, 171, 175, 183

156)
$$\dot{\begin{pmatrix} x \\ y \end{pmatrix}} = \begin{pmatrix} a(t) & b(t) \\ -b(t) & a(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Substitution:

$$x = r\cos\varphi \Rightarrow \dot{x} = \dot{r}\,\overset{\cos}{\cancel{\sin}}\,\varphi - r\dot{\varphi}\sin\varphi$$

$$y = r\sin\varphi \Rightarrow \dot{y} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi$$

Einsetzen:

$$\begin{cases} \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi = a(t)\,r\cos\varphi + b(t)\,r\sin\varphi \Leftrightarrow \dot{r} = a(t)r + b(t)r\tan\varphi + r\dot{\varphi}\tan\varphi \\ \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi = -b(t)\,r\cos\varphi + a(t)\,r\sin\varphi \end{cases}$$

$$\cancel{\dot{r} = a(t)r + b(t)r\tan\varphi + r\dot{\varphi}\tan\varphi}$$

$$\Rightarrow \cancel{r}\dot{\varphi}\tan\varphi\sin\varphi + b(t)\cancel{r}\tan\varphi\sin\varphi + \cancel{a(t)r\sin\varphi} + \cancel{r}\dot{\varphi}\cos\varphi = -b(t)\cancel{r}\cos\varphi + \cancel{a(t)r\sin\varphi}$$

$$\Leftrightarrow \dot{\varphi}(\tan\varphi\sin\varphi + \cos\varphi) + b(t)(\tan\varphi\sin\varphi + \cos\varphi) = 0$$

$$\Leftrightarrow \dot{\varphi} = -b(t) \Rightarrow \varphi = -\int b(t)\,dt$$

$$\Rightarrow \dot{r} = a(t)r + b(t)r\tan\varphi - b(t)r\tan\varphi = a(t)r$$

$$\Leftrightarrow \frac{\dot{r}}{r} = a(t) \Rightarrow r = e^{\int a(t)\,dt}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = e^{\int a(t)\,dt} \begin{pmatrix} \cos\left(-\int b(t)\,dt\right) & -\sin\left(-\int b(t)\,dt\right) \\ \sin\left(-\int b(t)\,dt\right) & \cos\left(-\int b(t)\,dt\right) \end{pmatrix}$$

Lösungsbasis, da $W(t) = r\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \Rightarrow D(t) = r^2(\cos^2\varphi + \sin^2\varphi) = r^2 \neq 0.$

157)
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 3t^2 & -t \\ t & 3t^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t e^{t^3} \\ t e^{t^3} \end{pmatrix}$$

Notation wie Bsp 156):

$$\int a(t)\,dt = 3\int t^2\,dt = t^3$$

$$-\int b(t)\,dt = \int t\,dt = \frac{t^2}{2}$$

$\Rightarrow$ Lösungen des <u>homogenen Systems:</u>

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{t^3}\cos\frac{t^2}{2} & -e^{t^3}\sin\frac{t^2}{2} \\ e^{t^3}\sin\frac{t^2}{2} & e^{t^3}\cos\frac{t^2}{2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Variation der Konstanten:

$$c(t) = \int W^{-1}(t)\, b(t)$$

$$= \int \cancel{e^{-t^3}} \begin{pmatrix} \cos\frac{t^2}{2} & \sin\frac{t^2}{2} \\ -\sin\frac{t^2}{2} & \cos\frac{t^2}{2} \end{pmatrix} \cancel{e^{t^3}} \begin{pmatrix} t \\ t \end{pmatrix} dt$$

$$\int t\cos\frac{t^2}{2} \overset{dt}{=} \left| \begin{matrix} \frac{t^2}{2} = u \\ t\,dt = du \end{matrix} \right| = \int \cos u\,du = \sin\frac{t^2}{2}$$

$$\Rightarrow c(t) = \begin{pmatrix} \sin\frac{t^2}{2} - \cos\frac{t^2}{2} \\ \cos\frac{t^2}{2} + \sin\frac{t^2}{2} \end{pmatrix}$$

<u>partikuläre Lösung:</u>

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{t^3} \begin{pmatrix} \cos\frac{t^2}{2} & -\sin\frac{t^2}{2} \\ +\sin\frac{t^2}{2} & \cos\frac{t^2}{2} \end{pmatrix} \begin{pmatrix} \sin\frac{t^2}{2} - \cos\frac{t^2}{2} \\ \cos\frac{t^2}{2} + \sin\frac{t^2}{2} \end{pmatrix} = e^{t^3} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$\Rightarrow$ <u>allgemeine Lösung:</u>

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{t^3}\cos\frac{t^2}{2} & -e^{t^3}\sin\frac{t^2}{2} \\ e^{t^3}\sin\frac{t^2}{2} & e^{t^3}\cos\frac{t^2}{2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} -e^{t^3} \\ e^{t^3} \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} x = e^{t^3}\left(\cos\frac{t^2}{2} c_1 - \sin\frac{t^2}{2}\cdot c_2 - 1\right) \\ y = e^{t^3}\left(\sin\frac{t^2}{2} c_1 + \cos\frac{t^2}{2}\cdot c_2 + 1\right) \end{cases}$$

164) $\begin{cases} t^2 \dot{x} = -2y + t \\ \dot{y} = -x + 1 \end{cases}$

Potenzreihenansatz:

$$x(t) = \sum_{n=0}^{\infty} a_n t^n \quad , \quad y(t) = \sum_{n=0}^{\infty} b_n t^n \quad \text{mit } a_0 = x(0) = 1, \; b_0 = y(0) = 0$$

Einsetzen:

$$\begin{cases} \sum_{n=1}^{\infty} n a_n t^{n+1} = -2 \sum_{n=0}^{\infty} b_n t^n + t \\ \sum_{n=1}^{\infty} n b_n t^{n-1} = -\sum_{n=0}^{\infty} a_n t^n + 1 \end{cases}$$

Koeffizientenvergleich:

$g=0$: $\begin{cases} 0 = -2b_0 \\ b_1 = -a_0 + 1 \Rightarrow b_1 = 0 \end{cases}$ $\lightning$ $\Rightarrow$ Korrektur des AWP: $x(0) = \frac{1}{2} \Rightarrow a_0 = \frac{1}{2}$, $b_1 = \frac{1}{2}$

$g=1$: $\begin{cases} 0 = -2b_1 + 1 \Rightarrow b_1 = \frac{1}{2} \\ 2b_2 = -a_1 \end{cases}$

$\Rightarrow b_2 = c, \; a_1 = -2c$

$g=2$: $\begin{cases} a_1 = -2b_2 \\ 3b_3 = -a_2 \end{cases}$

Für $k \geq 2$ zerfällt die Koeff-Bestimmung in autonome GS:

$$\begin{cases} k\, b_k = -a_{k-1} \\ (k-1)\, a_{k-1} = -2b_k \end{cases} \Rightarrow k(k-1)\, b_k = 2b_k \Leftrightarrow k = 2 \;\vee\; b_k = 0 \Downarrow a_{k-1} = 0$$

$\Rightarrow$ Potenzreihenlösung: $x = \frac{1}{2} - 2ct$

$y = \frac{1}{2} t + c t^2$

168) $t \ddot{x} - (t+1)\dot{x} - 2(t-1)x = 0$

Exponentiallösung:

Ansatz: $x = e^{at} \Rightarrow \dot{x} = a e^{at}, \; \ddot{x} = a^2 e^{at}$

$\rightsquigarrow a^2 t e^{at} - (t+1) a e^{at} - 2(t-1) e^{at} = 0$

$\Leftrightarrow t\underbrace{(a^2 - a - 2)}_{\text{muss 0 sein, da } a \text{ nicht von } t \text{ abh. soll}} = a - 2 \Rightarrow a = 2$

$\Rightarrow \varphi(t) = \cancel{t} \cdot e^{2t}$

<u>Reduktionsverfahren:</u>

$$\psi(t) = \sigma(t)\cdot\varphi(t) \Rightarrow \dot\psi(t) = \dot\sigma(t)\,e^{2t} + 2\,\sigma(t)\,e^{2t}$$

$$\Rightarrow \ddot\psi(t) = \ddot\sigma(t)\,e^{2t} + 2\dot\sigma(t)e^{2t} + 2\dot\sigma(t)e^{2t} + 4\sigma(t)\,e^{2t}$$

Einsetzen:

$$t\,(\ddot\sigma(t) + 4\dot\sigma(t) + 4\sigma(t)) - (t+1)(\dot\sigma(t) + 2\sigma(t)) - 2(t-1)\sigma(t) = 0$$

$$\Leftrightarrow t\,\ddot\sigma(t) + \dot\sigma(t)(4t - t - 1) + \cancel{\sigma(t)(4t - 2t - 2 - 2t + 2)} = 0$$

$$\tau(t) := \dot\sigma(t) \Rightarrow t\dot\tau(t) + \tau(t)(3t-1) = 0$$

$$\Rightarrow \int \frac{\dot\tau(t)}{\tau(t)}\,dt = \int \frac{1-3t}{t}\,dt \Leftrightarrow \ln|\tau(t)| = \ln|t| - 3t$$

$$\Rightarrow \tau(t) = t\,e^{-3t}$$

$$\Rightarrow \sigma(t) = \int \tau(t)\,dt = \int \underset{g}{t}\,\underset{f'}{e^{-3t}}\,dt = -\frac{t}{3}e^{-3t} + \frac{1}{3}\int e^{-3t}\,dt$$

$$= -e^{-3t}\left(\frac{t}{3} + \frac{1}{9}\right)$$

$$\Rightarrow \psi(t) = -e^{-3t}\left(\frac{t}{3} + \frac{1}{9}\right)e^{2t} = -\frac{1}{9}\left(e^{-t}(3t+1)\right)$$

$\Rightarrow$ <u>allgemeine Lösung:</u>

$$x = \begin{pmatrix} e^{2t} & e^{-t}(3t+1) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

171) $(1+t^2)\ddot x - 2t\dot x + 2x = 0$

<u>Potenzreihenansatz:</u>

$$x(t) = \sum_{n=0}^{\infty} a_n t^n \quad \text{mit} \quad a_0 = x(0) = c_1,\ a_1 = \dot x(0) = c_2$$

Einsetzen:

$$(1+t^2)\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - 2t\sum_{n=1}^{\infty} n a_n t^{n-1} + 2\sum_{n=0}^{\infty} a_n t^n = 0$$

$$\Leftrightarrow \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n t^n - 2\sum_{n=1}^{\infty} n a_n t^n + 2\sum_{n=0}^{\infty} a_n t^n = 0$$

$$\Leftrightarrow \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n + \sum_{n=2}^{\infty} \underbrace{(n(n-1) - 2n + 2)}_{(n-2)(n-1)} a_n t^n - \cancel{2a_1 t} + \cancel{2a_1 t} + 2a_0 = 0$$

<u>Koeff.-Vgl.:</u> $q=0$: $\quad 2a_2 + 2a_0 = 0 \Rightarrow a_2 = -a_0 = -c_1$

$q=1$: $\quad 6a_3 = 0 \Rightarrow a_3 = 0$

$k := g > 2: \quad (k+2)(k+1)\,a_{k+2} + (k-2)(k-1)\,a_k = 0$

$a_3 = a_4 = 0 \quad \Rightarrow \quad a_k = 0 \quad \forall k \geq 5$

$\Rightarrow$ Lösung des AWP $(x, \dot{x})(0) = (c_1, c_2)$: $\quad x = c_1 + c_2 t - c_1 t^2$

$c_1 = 1,\ c_2 = -1 \quad \rightsquigarrow \quad x = 1 - t - t^2.$

175)
$$A = \begin{pmatrix} 2 & 13 & 1 \\ -1 & 5 & 1 \\ 3 & 0 & 0 \end{pmatrix}$$

$$\chi_A(\lambda) = \begin{vmatrix} 2-\lambda & 13 & 1 \\ -1 & 5-\lambda & 1 \\ 3 & 0 & -\lambda \end{vmatrix} = (2-\lambda)(5-\lambda)(-\lambda) + 39 - 3(5-\lambda) - 13\lambda$$
$$= -\lambda^3 + 7\lambda^2 - 20\lambda + 24 = 0$$

Suche nach ganzzahligen Nullstellen mit dem Horner-Schema:

	1	-7	20	-24
1	1	-6	14	-10
2	1	-5	10	-4
3	1	-4	8	0 ✓

$\Rightarrow \chi_A(\lambda) = (-1)(\lambda - 3)(\lambda^2 - 4\lambda + 8) \quad \Rightarrow \quad \lambda_1 = 3,\ \lambda_2 = 2+2i,\ \lambda_3 = 2-2i$

$\Rightarrow J = \operatorname{diag}(3, 2+2i, 2-2i) \quad \Rightarrow \quad e^{tJ} = \operatorname{diag}(\cancel{3}e^{3t}, e^{2t(1+i)}, e^{2t(1-i)})$

e^{tJ} in reeller Form:

$$\begin{pmatrix} \operatorname{Re} e^{tJ} & -\operatorname{Im} e^{tJ} \\ \operatorname{Im} e^{tJ} & \operatorname{Re} e^{tJ} \end{pmatrix} = \left(\begin{array}{ccc|ccc} e^{3t} & e^{2t}\cos 2t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -e^{2t}\sin 2t & 0 \\ 0 & 0 & e^{2t}\cos 2t & 0 & 0 & e^{2t}\sin 2t \\ \hline 0 & 0 & 0 & e^{3t} & 0 & 0 \\ 0 & e^{2t}\sin 2t & 0 & 0 & e^{2t}\cos 2t & 0 \\ 0 & 0 & -e^{2t}\sin 2t & 0 & 0 & e^{2t}\cos 2t \end{array}\right)$$

wegen $e^{a+bi} = e^a e^{ib} = e^a(\cos b + i \sin b) = e^a \cos b + i e^a \sin b$.

183)
$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 2 & 13 & 1 \\ -1 & 5 & 1 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Basis aus Eigenvektoren:

$\lambda_1 = 3$:
$$\begin{pmatrix} -1 & 13 & 1 \\ -1 & 2 & 1 \\ 3 & 0 & -3 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda_2 = 2+2i$
$$\begin{pmatrix} -2i & 13 & 1 \\ -1 & 3-2i & 1 \\ 3 & 0 & -2-2i \end{pmatrix}$$

Betrachte 3. Zeile $\Rightarrow$ Gestalt von $v_2 = c\,(2+2i,\ v_{22},\ 3)$

2. Zeile $\rightsquigarrow$ $-2-2i + v_{22}(3-2i) + 3 = 0 \iff \underbrace{(3-2i)}_{=:c} v_{22} = 2i-1$

$$\Rightarrow v_2 = \begin{pmatrix} 10+2i \\ 2i-1 \\ 9-6i \end{pmatrix}, \quad v_3 = \overline{v_2}$$

$$\Rightarrow T = \begin{pmatrix} 1 & 10+2i & 10-2i \\ 0 & 2i-1 & -1-2i \\ 1 & 9-6i & 9+6i \end{pmatrix}$$

$\Rightarrow$ komplexe Lösungsbasis:

$$T \cdot e^{\lambda t} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{3t}, \begin{pmatrix} 10+2i \\ 2i-1 \\ 9-6i \end{pmatrix} e^{2t+2it}, \begin{pmatrix} 10-2i \\ -1-2i \\ 9+6i \end{pmatrix} e^{2t-2it}$$

$\Rightarrow$ reelle Lösungsbasis:

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{3t}, \begin{pmatrix} 10 \\ -1 \\ 9 \end{pmatrix} e^{2t} \cos 2t - \begin{pmatrix} 2 \\ 2 \\ -6 \end{pmatrix} e^{2t} \sin 2t, \begin{pmatrix} 10 \\ -1 \\ 9 \end{pmatrix} e^{2t} \cos 2t + \begin{pmatrix} 2 \\ 2 \\ -6 \end{pmatrix} e^{2t} \sin 2t$$

wegen $b\, e^{2t+2it} = b\, e^{2t} \underbrace{e^{2it}}_{=\cos 2t + i \sin 2t} = \mathrm{Re}\, b \cdot \cos 2t\, e^{2t} + i\, \mathrm{Im}\, b\, e^{2t} \sin 2t$