Angewandte Statistik UE

V) 1.a) $X_1, \dots, X_n$ Stichprobe; $X \sim A_\theta$

ges.: (i) FRC-Schranke, (ii) FRC-effizienter Schätzer für θ.

(i) Sei T eine unverzerrte Schätzfkt. für θ, d.g.

$$\mathrm{Var}_\theta T \geq \frac{[\overbrace{\frac{\partial}{\partial\theta}\theta}^{1}]^2}{n \cdot \mathbb{E}_\theta[\frac{\partial}{\partial\theta}\ln P(X|\theta)]^2} =: FRC$$

$$X \sim A_\theta \Rightarrow P([X=x|\theta]) = \theta^x(1-\theta)^{1-x} \quad (x=0,1) \qquad \theta \in (0,1).$$

$$\mathbb{E}_\theta X = \theta, \quad \mathrm{Var}_\theta X = (1-\theta)\theta. \Rightarrow \mathbb{E}_\theta X^2 = \theta.$$

$$\mathbb{E}_\theta[\frac{\partial}{\partial\theta}\ln P(X|\theta)]^2 = \sum_{x=0}^{1}[\frac{\partial}{\partial\theta}\underbrace{\ln(\theta^x(1-\theta)^{1-x})}_{x\ln\theta+(1-x)\ln(1-\theta)}]^2\,\theta^x(1-\theta)^{1-x}$$

$$= \left(-\frac{1}{1-\theta}\right)^2(1-\theta) + \frac{1}{\theta^2}\theta = \frac{1}{1-\theta} + \frac{1}{\theta} = \frac{1}{\theta(1-\theta)}$$

$$\Rightarrow FRC = \frac{\theta(1-\theta)}{n}$$

(ii) Sei $T = t(X_1, \dots, X_n) = \overline{X}_n$.

$$\mathbb{E}_\theta(T) = \mathbb{E}_\theta\left(\frac{1}{n}\sum_{i=1}^n X_i\right) = \frac{1}{n}\, n\, \mathbb{E}_\theta X = \theta$$

$$\mathbb{E}_\theta T^2 = \mathbb{E}_\theta\left(\frac{1}{n^2}\underbrace{\left(\sum_{i=1}^n X_i\right)^2}_{\sum_{i=1}^n X_i^2 + \sum_{i,j=1, i\neq j}^n X_i X_j}\right) = \frac{1}{n^2}\left(\sum_{i=1}^n \mathbb{E}X_i^2 + \sum_{i,j=1}^n \underbrace{\mathbb{E}X_i\,\mathbb{E}X_j}_{(\mathbb{E}X)^2}\right)$$

$$= \frac{1}{n^2}\left(n\underbrace{\mathbb{E}X^2}_{\theta} + n(n-1)\underbrace{(\mathbb{E}X)^2}_{\theta^2}\right) = \frac{1}{n}\left(\theta + (n-1)\theta^2\right)$$

$$\Rightarrow \underset{\frac{1}{n}\mathrm{Var}X}{\overset{}{\mathrm{Var}\,T}} = \frac{\theta}{n} + \frac{n-1}{n}\theta^2 - \theta^2 = \frac{\theta-\theta^2}{n} = \frac{\theta(1-\theta)}{n} \quad \Rightarrow \text{FRC-effizient}$$

$$\mathrm{Var}\,T = \mathrm{Var}\,\overline{X}_n = \mathrm{Var}\,\frac{1}{n}\sum_{i=1}^n X_i = \frac{1}{n^2}\mathrm{Var}\sum_{i=1}^n X_i = \frac{1}{n^2}\, n \sum \mathrm{Var}\,X_i = \frac{1}{n}\mathrm{Var}\,X.$$

b) selbe Diskussion für $X \sim P_\mu$.

$$X \sim P_\mu \Rightarrow P([X=x \mid \mu]) = \frac{\mu^x}{x!} e^{-\mu} \quad x \in \mathbb{N}_0 \;,\; \mu \in (0,\infty)$$

$$\Rightarrow \mathbb{E}X = \operatorname{Var} X = \mu \Rightarrow \mathbb{E}X^2 = \mu + \mu^2 = \mu(1+\mu)$$

(i) $$E_\mu \left[\frac{\partial}{\partial \mu} \ln P(X \mid \mu) \right]^2 = \sum_{n=0}^{\infty} \Big[\frac{\partial}{\partial \mu} \underbrace{\ln\left(\frac{\mu^n}{n!} e^{-\mu} \right)}_{n \ln \mu - \ln n! - \mu} \Big]^2 \frac{\mu^n}{n!} e^{-\mu}$$

$$= \sum_{n=0}^{\infty} \left(\frac{n}{\mu} - 1 \right)^2 \frac{\mu^n}{n!} e^{-\mu} = \mathbb{E} \underbrace{\left(\frac{X}{\mu} - 1 \right)^2}_{\frac{X^2}{\mu^2} - 2\frac{X}{\mu} + 1} = \frac{\mathbb{E}X^2}{\mu^2} - 2 \frac{\mathbb{E}X}{\mu} + 1 = \frac{1+\mu}{\mu} - 1 = \frac{1}{\mu}$$

$$\Rightarrow FRC = \frac{\mu}{n}.$$

(ii) Sei $T := \overline{X}_n$, d.g.

$$\mathbb{E}_\theta T = \mathbb{E}X = \mu$$

$$\mathbb{E}_\theta T^2 = \frac{1}{n} \Big(\underbrace{\mathbb{E}X^2}_{\mu + \mu^2} + (n-1) \underbrace{(\mathbb{E}X)^2}_{\mu^2} \Big) = \frac{\mu}{n} + \mu^2$$

$$\Rightarrow \underset{\frac{1}{n}\operatorname{Var} X}{\underset{\shortparallel}{\operatorname{Var} T}} = \frac{\mu}{n}.$$

2) $\underline{\text{ZZ:}}$ $$\mathbb{E}_\theta \left[\frac{\partial \ln f(X \mid \theta)}{\partial \theta} \right]^2 = - \mathbb{E}_\theta \left[\frac{\partial^2 \ln f(X \mid \theta)}{\partial \theta^2} \right]$$

$$\int \left[\frac{\partial \ln f(x \mid \theta)}{\partial \theta} \right]^2 f(x \mid \theta) \, d\lambda(x) + \int \left[\frac{\partial^2 \ln f(x \mid \theta)}{\partial \theta^2} \right] f(x \mid \theta) \, d\lambda(x) =$$

$$= \int \left(\frac{1}{f(x \mid \theta)} \frac{\partial f(x \mid \theta)}{\partial \theta} \right)^2 f(x \mid \theta) \, d\lambda(x) + \int \frac{\partial}{\partial \theta} \left[\frac{1}{f(x \mid \theta)} \frac{\partial f(x \mid \theta)}{\partial \theta} \right] f(x \mid \theta) \, d\lambda(x)$$

$$= \underbrace{\int \frac{1}{f(x \mid \theta)} \left(\frac{\partial f(x \mid \theta)}{\partial \theta} \right)^2 d\lambda(x) + \int \left(- \frac{1}{f(x \mid \theta)^2} \left(\frac{\partial f(x \mid \theta)}{\partial \theta} \right)^2 + \frac{1}{\cancel{f(x \mid \theta)}} \frac{\partial^2 f(x \mid \theta)}{\partial \theta^2} \right) \cancel{f(x \mid \theta)} \, d\lambda(x)}_{=0}$$

$$= \int \frac{\partial^2 f(x \mid \theta)}{\partial \theta^2} = \frac{\partial^2}{\partial \theta^2} \underbrace{\int f(x \mid \theta)}_{1} = 0.$$

1a) $$\mathbb{E}_\theta \left[\frac{\partial \ln P(X \mid \theta)}{\partial \theta} \right]^2 = \frac{1}{\theta(1-\theta)} \qquad \mathbb{E}_\theta \Big[\overbrace{\frac{\partial^2 \ln P(X \mid \theta)}{\partial \theta^2}}^{\frac{\partial}{\partial \theta}\left(\frac{X}{\theta} - \frac{1-X}{1-\theta} \right)} \Big] = - \frac{\theta}{\theta^2} - \frac{1-\theta}{(1-\theta)^2} = - \left(\frac{1}{\theta} + \frac{1}{1-\theta} \right)$$

$$\mathbb{E}_\mu \left[\frac{\partial \ln P(X \mid \mu)}{\partial \mu} \right]^2 = \frac{1}{\mu} \qquad \mathbb{E}_\mu \Big[\underbrace{\frac{\partial^2 \ln P(X \mid \mu)}{\partial \mu^2}}_{\frac{\partial}{\partial \mu}\left(\frac{X}{\mu} - 1 \right)} \Big] = \mathbb{E}_\mu \left(- \frac{n}{\mu^2} \right) = - \frac{\mu}{\mu^2} = - \frac{1}{\mu}.$$

4) $X_1, \dots, X_n$ Stichprobe; $X \sim N(\mu, \sigma^2)$ [μ bekannt]

Schätzer für σ^2:

$$S_1 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)^2 \; ; \qquad S_2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2$$

a) Schätzer unverzerrt?

•) $\mathbb{E} S_2 = \sigma^2$ (siehe VO)

•) ~~Wissen: $\frac{n-1}{n} \frac{S_2}{\sigma^2} = \frac{S_1}{\sigma^2} \sim \chi^2_n$~~

Satz 1.3.3: $\underbrace{\frac{n}{\sigma^2} S_1}_{=: Y_n} \sim \chi^2_n$.

$$\Rightarrow \mathbb{E} S_1 = \frac{\sigma^2}{n} \mathbb{E} Y_n = \frac{\sigma^2}{n} n = \sigma^2$$

b) Welcher Schätzer ist effizienter?

•) $\mathrm{Var}\, S_2 \overset{\text{UE3, Bsp 3}}{=} \frac{2\sigma^4}{n-1}$

•) $\mathrm{Var}\, S_1 = \frac{\sigma^4}{n^2} \mathrm{Var}\, Y_n = \frac{\sigma^4}{n^2} 2n = 2 \frac{\sigma^4}{n}$

$\Rightarrow \mathrm{Var}\, S_1 < \mathrm{Var}\, S_2$

$\Rightarrow S_1$ effizienter

c) Schätzer konsistent?

erwartungstreu, asymptotisch effizient $\Rightarrow$ konsistent

d) ZZ: S_1 ist CAN-Schätzfolge

1) S_1 konsistent

2) $\underline{\text{ZZ}}$: $\sqrt{n}\,(S_{1n} - \sigma^2) \xrightarrow{Vt} N(0, \psi(\sigma^2))$

$$K_i := \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2_1 \quad \Rightarrow \quad \mathbb{E} K_i = 1, \; \mathrm{Var}\, K_i = 2$$

$$S_1 = \sigma^2 \cdot \overline{K_n} = \sigma^2 \cdot \frac{1}{n} \sum_{i=1}^{n} K_i$$

$$\text{ZGV} \Rightarrow \sqrt{n} \, \frac{\overline{K_n} - 1}{\sqrt{2}} = \sqrt{n} \, \frac{\frac{S_1}{\sigma^2} - 1}{\sqrt{2}} = \sqrt{n} \, \frac{S_1 - \sigma^2}{\sqrt{2}\sigma} \sim N(0,1)$$

$$\Rightarrow \sqrt{n}\,(S_1 - \sigma^2) \sim N(0, 2\sigma).$$

5) a) $X \sim Gam(\alpha, \beta) \quad \alpha, \beta > 0$

$$f(x|\alpha,\beta) = \frac{x^{\alpha-1}}{\Gamma(\alpha)\,\beta^\alpha} e^{-\frac{x}{\beta}} \cdot 1\!\!1_{(0,\infty)}(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot 1\!\!1_{(0,\infty)}(x) \cdot \exp\left[(\alpha-1)\ln x - \frac{1}{\beta}x\right]$$

$$\overset{\text{Satz 2.7.1}}{\Longrightarrow} S = s(X_1,\dots,X_n) = \left(\sum_{i=1}^n \ln X_i \,,\, \sum_{i=1}^n X_i\right) \text{ suff. für } (\alpha,\beta).$$

b) $X \sim Be_1(\alpha,\beta) \quad \alpha,\beta > 0$

$$f(x|\alpha,\beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} 1\!\!1_{(0,1)}(x) = \frac{1}{B(\alpha,\beta)} 1\!\!1_{(0,1)}(x) \cdot \exp\left[(\alpha-1)\overbrace{\ln x + (\beta-1)\ln(1-x)}^{\text{l.u.}}\right]$$

$$\Rightarrow S = \left(\sum_{i=1}^n \ln X_i \,,\, \sum_{i=1}^n \ln(1-X_i)\right) \text{ suff. für } (\alpha,\beta).$$

c) $X \sim U_{(a,b)} \quad a < b$

$$f(x|a,b) = \frac{1}{b-a} 1\!\!1_{(a,b)}(x) \exp[\overset{\text{nicht l.u.!}}{\downarrow}{0}] \quad \Rightarrow \text{ keine Exponentialfamilie}$$

$$\ell(a,b;x_1,\dots,x_n) = \prod_{i=1}^n f(x|a,b) = \frac{1}{(b-a)^n} 1\!\!1_{(a,b)}(x_1) \cdot \ldots \cdot 1\!\!1_{(a,b)}(x_n)$$

Sei $S = s(X_1,\dots,X_n) = (X_{(1)}, X_{(n)})$

$$\Rightarrow \ell(a,b;x_1,\dots,x_n) = \underbrace{\frac{1}{(b-a)^n} 1\!\!1_{(a,b)}(\min_{1\le i\le n} x_i)\, 1\!\!1_{(a,b)}(\max_{1\le i\le n} x_i)}_{\varphi[s(x_1,\dots,x_n);a,b]} \cdot \underbrace{1}_{\psi(x_1,\dots,x_n)}.$$

6) $X_1,\dots,X_n$ Stichprobe, $X \sim N(\mu,\sigma^2)$

ZZ: $S_n^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X}_n)^2$ bzgl. qu. Verlustfkt. $L(T;\theta) = (T-\theta)^2$ unzulässig.

d.h. $\exists$ Schätzfkt. T mit $\mathbb{E}(L(T),\sigma^2) \le \mathbb{E}(L(S_n^2,\sigma^2)) \quad \forall \sigma^2 \in (0,\infty)$

◦) $\mathbb{E}(L(S_n^2,\sigma^2)) = \mathbb{E}(S_n^2 - \sigma^2)^2 = \operatorname{Var} S_n^2 = \frac{2\sigma^4}{n-1}$

◦) Sei $T := \frac{n-1}{n} S_n^2$, d.g.

$$\mathbb{E}\left(L\left(\tfrac{n-1}{n}S_n^2;\sigma^2\right)\right) = \mathbb{E}\left(\tfrac{n-1}{n}S_n^2 - \sigma^2\right)^2 = \left(\tfrac{n-1}{n}\right)^2 \underbrace{\mathbb{E}(S_n^2)^2}_{\left(\frac{2\sigma^4}{n-1}+\sigma^4\right)} - 2\,\tfrac{n-1}{n}\underbrace{\mathbb{E}S_n^2}_{\sigma^2}\,\sigma^2 + \sigma^4$$

$$= \frac{2\sigma^4}{n} + \frac{\sigma^4}{n^2} < \mathbb{E}(L(S_n^2,\sigma^2)) \quad \forall n \in \mathbb{N}.$$