Analysis UE

IV, 55, 59, 63, 69, 73, 77, 85

55, $$\int \frac{dx}{\sqrt{x}\,(\sqrt[3]{x}+1)^3} = \left| \begin{array}{l} u = x^{\frac{1}{3}} \\ du = \frac{1}{3} x^{-\frac{2}{3}} dx = \frac{1}{3u^2} dx \end{array} \right|$$

$$= \int \frac{3u^2}{u^{\frac{3}{2}}(u+1)^3}\, du$$

$$= 3 \int \frac{\sqrt{u}}{(u+1)^3}\, du$$

$$= \left| \begin{array}{l} \sqrt{u} = g(x) \\ \frac{1}{(u+1)^3} = f'(x) \end{array} \right|$$

$$= 3\left(-\frac{\sqrt{u}}{2(u+1)^2} + \int \frac{du}{2(u+1)^2 \cdot 2\sqrt{u}} \right) + c$$

$$= \frac{3}{2}\left(-\frac{\sqrt{u}}{(u+1)^2} + \frac{1}{2}\int \frac{du}{(u+1)^2 \sqrt{u}} \right) + c =: \triangle$$

$$\int \frac{du}{(u+1)^2\sqrt{u}} = \left| \begin{array}{l} t = \frac{1}{u+1} \\ dt = -t^2\, du \end{array} \right|$$

$$= \int \frac{t^2}{-t^2} \sqrt{\frac{t}{1-t}}\, dt = -\int \frac{\sqrt{t}}{\sqrt{1-t}}\, dt$$

$$= \left| \begin{array}{l} y = \sqrt{1-t} \\ dy = -\frac{1}{2y} \end{array} \right|$$

$$= \int \frac{\cancel{2y}\sqrt{1-y^2}}{\cancel{y}}\, dy = 2\int \sqrt{1-y^2}\, dy =: \circledast$$

2. Hermite-Ansatz:

$$\int \sqrt{1-y^2}\, dy = (q_0 + q_1 y)\sqrt{1-y^2} + R \int \frac{dy}{\sqrt{1-y^2}} + c$$

Koeff.-Vgl.: $1 - y^2 = q_1(1-y^2) - (q_0 + q_1 y) y + R$

$$\Rightarrow q_0 = 0, \quad q_1 = R = \frac{1}{2}$$

$$\Rightarrow (*) = y\sqrt{1-y^2} + \underbrace{\int \frac{dy}{\sqrt{1-y^2}}}_{\arcsin y} + c$$

$$= \sqrt{1-t}\sqrt{t} + \arcsin\sqrt{1-t} + c$$

$$= \sqrt{(1-t)t} + \arctan\sqrt{\frac{1-t}{t}} + c$$

$$= \sqrt{\left(1-\frac{1}{u+1}\right)\frac{1}{u+1}} + \arctan\sqrt{\frac{(u+1)\cdot u}{(u+1)}} + c$$

$$= \frac{\sqrt{u}}{u+1} + \arctan\sqrt{u} + c$$

$$\Rightarrow (\triangle) = -\frac{3\sqrt{u}}{2(u+1)^2} + \frac{3\sqrt{u}}{4(u+1)} + \frac{3}{4}\arctan\sqrt{u} + c$$

$$= \frac{3\sqrt{u}(u-1)}{4(u+1)^2} + \frac{3}{4}\arctan\sqrt{u} + c$$

$$= \frac{3\sqrt[6]{x}(\sqrt[3]{x}-1)}{4(\sqrt[3]{x}+1)^2} + \frac{3}{4}\arctan\sqrt[6]{x} + c$$

59) $$\int \frac{dx}{\sqrt[3]{\sin^4 x\cos^2 x}} = \left| \begin{array}{l} \tan x = t \\ (1+t^2)\,dx = dt \end{array} \right|$$

$$= \int \frac{dt}{\sqrt[3]{\frac{t^4}{(1+t^2)^2}\cdot\frac{1}{(1+t^2)}}\,(1+t^2)}$$

$$= \int \frac{dt}{\sqrt[3]{t^4}} = \left| \begin{array}{l} u = t^{-\frac{1}{3}} \\ du = -\frac{1}{3}t^{-\frac{4}{3}}dt \end{array} \right| = \int u^4\cdot(-3)\cdot u^{-4}\,du = -\int 3\,du$$

63) $\displaystyle\int \frac{\sqrt[4]{x^3}}{\sqrt[8]{(2x^2+1)^7}}\,dx = \left| \begin{array}{l} z = \sqrt{2}\,x \\ dz = \sqrt{2}\,dx \end{array} \right|$

$$= \int \frac{\left(\frac{z}{\sqrt{2}}\right)^{\frac{3}{4}}}{(z^2+1)^{\frac{7}{8}}\sqrt{2}}\,dz$$

$$= \frac{\sqrt[8]{2}}{2} \int \frac{z^{\frac{3}{4}}}{(z^2+1)^{\frac{7}{8}}}\,dz$$

$$= \frac{\sqrt[8]{2}}{2} \int \sqrt[4]{\frac{z^3}{\left(\sqrt{z^2+1}\right)^7}}\,dz$$

$$= \left| \begin{array}{l} z = \sinh t \\ dz = \cosh t\,dt \end{array} \right|$$

$$= \frac{\sqrt[8]{2}}{2} \int \sqrt[4]{\frac{\sinh^3 t}{\cosh^7 t}} \cdot \cosh t\,dt$$

$$= \frac{\sqrt[8]{2}}{2} \int \sqrt[4]{\tanh^3 t}\,dt$$

$$= \left| \begin{array}{l} u = (\tanh t)^{\frac{3}{4}} \\ du = \frac{3}{4}(\tanh t)^{-\frac{1}{4}}(1-\tanh^2 t)\,dt \end{array} \right|$$

$$= \frac{\sqrt[8]{2}}{2} \int u\, \frac{4}{3}\left(u^{\frac{4}{3}}\right)^{\frac{1}{4}} \frac{1}{1-u^{\frac{8}{3}}}\,du$$

$$= \frac{2\sqrt[8]{2}}{3} \int \frac{u^{\frac{4}{3}}}{1-u^{\frac{8}{3}}}\,du$$

$$= \left| \begin{array}{l} y = u^{\frac{1}{3}} \\ dy = \frac{1}{3}u^{-\frac{2}{3}}\,du \end{array} \right|$$

$$= \frac{2\sqrt[8]{2}}{\cancel{3}} \int \frac{y^4}{1-y^8}\,\cancel{3}y^2\,dy$$

$$= 2\sqrt[8]{2} \int \frac{y^6}{1-y^8}\,dy$$

69) $$\int \frac{(2x+1)\,dx}{(3x^2+4x+4)\sqrt{x^2+6x-1}} =: \circledast$$

$$x = \frac{\mu t + v}{t+1}$$

$$x^2 + \frac{4}{3}x + \frac{4}{3} = \frac{(\mu t + v)^2 + \frac{4}{3}(\mu t + v)(t+1) + \frac{4}{3}(t+1)^2}{(t+1)^2}$$

$$x^2 + 6x - 1 = \frac{(\mu t + v)^2 + 6(\mu t + v)(t+1) - (t+1)^2}{(t+1)^2}$$

Grad 1: $$\left.\begin{aligned} 2\mu v + \tfrac{4}{3}(\mu + v) + \tfrac{8}{3} &= 0 \\ 2\mu v + 6(\mu + v) - 2 &= 0 \end{aligned}\right\} \Rightarrow \mu = 2,\ v = -1$$

$$\Rightarrow \circledast = \left| \begin{aligned} x &= \frac{2t-1}{t+1} \\ dx &= \frac{3}{(t+1)^2}\,dt \end{aligned} \right|$$

$$= \frac{3}{3} \cdot \int \frac{\frac{4t-2+t+1}{t+1}}{\left(\frac{4t^2-4t+1}{(t+1)^2} + \frac{8t-4}{3(t+1)} + \frac{4}{3}\right)\sqrt{\frac{4t^2-4t+1}{(t+1)^2} + \frac{12t-6}{t+1} - 1}} \frac{3}{(t+1)^2}$$

$$= 9 \cdot \int \frac{\frac{5t-1}{\cancel{t+1}}}{\frac{(24t^2+3)}{3\cancel{(t+1)^2}} \frac{\sqrt{15t^2-6}}{\cancel{(t+1)}} \cancel{(t+1)^2}}\,dt$$

$$= \int \frac{135t - 27}{(24t^2+3)\sqrt{15t^2-6}}\,dt$$

77) $$\int \frac{(x-5)\,dx}{(x^2+8)\sqrt{-2x^2+5}} = \int \frac{x\,dx}{(x^2+8)\sqrt{-2x^2+5}} - 5\int \frac{dx}{(x^2+8)\sqrt{-2x^2+5}}$$

$$\int \frac{x\,dx}{(x^2+8)\sqrt{-2x^2+5}} = \left| \begin{aligned} x^2+8 &= u \\ 2x\,dx &= du \end{aligned} \right|$$

$$= \frac{1}{2}\int \frac{du}{u\sqrt{21-2u}}$$

$$= \left| \begin{aligned} y &= \frac{1}{u} \\ dy &= -\frac{1}{u^2}\,du = -y^2\,du \end{aligned} \right|$$

$$= \frac{1}{2}\int \frac{1}{\cancel{\frac{1}{y}}\sqrt{21-\frac{2}{y}}}\left(-y^{\cancel{2}\,-1}\right)dy$$

$$= -\frac{1}{2}\int \frac{dy}{\sqrt{y^2\left(21-\frac{2}{y}\right)}} = -\frac{1}{2}\int \frac{dy}{\sqrt{21y^2-2y}}$$

weiter mit 2. Hermite-Ansatz

$$\int \frac{dx}{(x^2+8)\sqrt{-2x^2+5}} = \left| \begin{aligned} u^2 &= -2 + \frac{5}{x^2} \\ 2u\,du &= -\frac{10}{x^3}\,dx \end{aligned} \right|$$

$$= \int \frac{2u\sqrt{\left(\frac{5}{u^2+2}\right)^3}}{\left(\frac{5}{u^2+2}+8\right)\sqrt{-2\frac{5}{u^2+2}+5}\,(-10)}\,du$$

$$= \int \frac{\cancel{2}u\,\cancel{\left(\frac{5}{u^2+2}\right)}\sqrt{\cancel{\frac{5}{u^2+2}}}}{\frac{+8u^2+21}{\cancel{u^2+2}}\sqrt{\frac{\cancel{5}u^2}{\cancel{u^2+2}}}\,\cancel{(-10)}_{(-1)}}\,du$$

$$= -\int \frac{du}{8u^2+21}$$

$$\Rightarrow \int \frac{(x-5)\,dx}{(x^2+8)\sqrt{-2x^2+5}} = -\frac{1}{2}\int \frac{dy}{\sqrt{21y^2-2y}} - \int \frac{du}{8u^2+21}$$

85) $$\sum_{n=0}^{\infty} \frac{(-1)^n}{an+b} = \lim_{x\to 1^-} \sum_{n=0}^{\infty} \frac{(-1)^n}{an+b} x^{an+b}$$

$$= \lim_{c\to 1^-} \sum_{n=0}^{\infty} \frac{(-1)^n}{an+b} x^{an+b} \Big|_0^c$$

$$= \lim_{c\to 1^-} \int_0^c \sum_{n=0}^{\infty} (-1)^n x^{an+b-1}\,dx$$

$$= \lim_{c\to 1^-} \int_0^c x^{b-1} \sum_{n=0}^{\infty} (-x^a)^n\,dx$$

$$= \lim_{c\to 1^-} \int_0^c \frac{x^{b-1}}{1+x^a}\,dx = \int_0^1 \frac{x^{b-1}}{1+x^a}\,dx$$

Gleichung korrekt, da $\sum_{n=0}^{\infty}(-x^a)^n$ gleichm. konvergent auf $x \in (-1,1)$

$\Rightarrow$ intbar auf $[0,x]$ $\Rightarrow \exists \lim_{c\to 1^-} \int \sum_{n=0}^{\infty} (-x^a)^n\,dx$

Grenzfkt. v. $\sum_{n=0}^{\infty} \frac{(-1)^n}{an+b}$ $\exists$ lt. Leibnitz.

$$\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{4n+3} = \int_0^1 \frac{x^2}{1+x^4}\,dx$$

$$= \left| \begin{aligned} z &= x^{-1} \\ dz &= -z^2\,dx \end{aligned} \right|$$

$$= \int_0^1 \frac{z^{-2}}{1+z^{-4}} \cdot (-z^{-2})\,dz$$

$$= -\int_0^1 \frac{\frac{1}{\cancel{z^4}}}{\frac{z^4+1}{\cancel{z^4}}}\,dz = -\int_0^1 \frac{1}{z^4+1}\,dz$$

Partialbruchzerlegung:

$$z^4+1 = \left(z^2+\sqrt{2}z+1\right)\left(z^2-\sqrt{2}z+1\right)$$

$$\frac{1}{z^4+1} = \frac{Az+B}{z^2+\sqrt{2}z+1} + \frac{Cz+D}{z^2-\sqrt{2}z+1}$$

$$1 = (z^2-\sqrt{2}z+1)(Az+B)+(z^2+\sqrt{2}z+1)(Cz+D)$$

$$\begin{aligned} z^3&:\ 0 = A+C \\ z^2&:\ 0 = B-\sqrt{2}A+D+\sqrt{2}C \\ z&:\ 0 = -\sqrt{2}B+A+\sqrt{2}D+C \\ z^0&:\ 1 = B+D \end{aligned}$$

$$\Rightarrow A = \frac{\sqrt{2}}{4},\quad C = -\frac{\sqrt{2}}{4},\quad B = D = \frac{1}{2}$$

$$\Rightarrow \int \frac{dz}{z^4+1} = \int \frac{\frac{\sqrt{2}}{2}z+1}{2z^2+2\sqrt{2}z+2} \mp \int \frac{\mp\frac{\sqrt{2}}{2}z\mp 1}{2z^2-2\sqrt{2}z+2}$$

$$\int \frac{\pm\frac{\sqrt{2}}{2}z+1}{2z^2+2\sqrt{2}z+2} = \int \frac{\pm\sqrt{2}z+2+8z+4\sqrt{2}-8z-4\sqrt{2}}{2\left((\sqrt{2}z\pm1)^2+1\right)}\,dz$$

$$= \underbrace{\frac{1}{2}\int \frac{8z\pm4\sqrt{2}}{2z^2\pm2\sqrt{2}z+1}\,dz}_{=:A} + \underbrace{\frac{1}{2}\int \frac{\pm\sqrt{2}z+2-8z\mp4\sqrt{2}}{2\left((\sqrt{2}z\pm1)^2+1\right)}\,dz}_{=:B}$$

$$A = \int \frac{2z\pm\sqrt{2}}{z^2\pm\sqrt{2}z+1} = \left|\begin{matrix} z^2\pm\sqrt{2}z+1 = u \\ (2z\pm\sqrt{2})\,dz = du \end{matrix}\right| = \int\frac{1}{u} = \ln|u| \overset{+c}{=} \ln\left|x^2\pm\sqrt{2}x+1\right| + c$$

$$B = \left|\begin{matrix} y = \sqrt{2}z\pm1 \\ dy = \sqrt{2}\,dz \end{matrix}\right| = \frac{1}{2}\int \frac{y(\pm1-4\sqrt{2})+1}{(y^2+1)\sqrt{2}}\,dy$$

$$= \frac{\pm1-4\sqrt{2}}{4\sqrt{2}}\int\frac{2y}{y^2+1}\,dy + \frac{1}{2\sqrt{2}}\int\frac{1}{y^2+1}\,dy$$

$$= \left(\pm\frac{\sqrt{2}}{8}-1\right)\ln|y^2+1| + \frac{\sqrt{2}}{4}\arctan y + c$$

$$\Rightarrow -\int_0^1 \frac{dz}{z^4+1} = -\ln|z^2+\sqrt{2}z+1| - \left(\frac{\sqrt{2}}{8}-1\right)\overbrace{\ln|2z^2+2\sqrt{2}z+2|}^{=\ln2+\ln|z^2+\sqrt{2}z+1|} - \frac{\sqrt{2}}{4}\arctan(\sqrt{2}z+1)$$

$$- \ln|z^2-\sqrt{2}z+1| + \left(\frac{\sqrt{2}}{8}+1\right)\ln|2z^2-2\sqrt{2}z+2| - \frac{\sqrt{2}}{4}\arctan(\sqrt{2}z-1)\Big|_0^1$$

$$= \overbrace{\frac{\sqrt{2}}{8}\ln1}^{0} - 2\ln2 + \frac{\sqrt{2}}{4}\overbrace{\left(\arctan(-1)+\arctan(1)\right)}^{0}$$

$$+ \frac{\sqrt{2}}{8}\underbrace{\ln\left|\frac{1+\sqrt{2}+1}{1-\sqrt{2}+1}\right|}_{=2\ln(\sqrt{2}+1)} + 2\ln2 - \frac{\sqrt{2}}{4}\left(\underbrace{\arctan(\sqrt{2}-1)+\arctan(\sqrt{2}+1)}_{=\frac{\pi}{2}}\right)$$

$$= +\frac{\sqrt{2}}{4}\left(\ln(\sqrt{2}+1) - \frac{\pi}{2}\right)$$