TRANSPORTATION AND PATTERNS OF URBAN DEVELOPMENT

AN AGGREGATIVE MODEL OF RESOURCE ALLOCATION IN A METROPOLITAN AREA*

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I. *Introduction*

The purpose of this paper is to put forth a simplified, aggregative model that will help to explain the sizes and structures of urban areas. The viewpoint taken is that the basic characteristics of cities are to be understood as market responses to opportunities for production and income. Properties of production functions are at the heart of the explanation of city size and structure in the model developed here, in much the way that properties of production functions are at the heart of modern neoclassical growth theory.

The general ideas that motivate the selection of the model developed below are commonplace in the voluminous recent literature on urban economics and geography. It has frequently been observed that the large size and rapid recent growth of urban areas are responses to income and employment opportunities provided there. It is but a small step from this observation to the assumption that the conditions of production differ in crucial respects as between urban and non-urban areas and as between urban areas of different size. Likewise, it is a common observation on the structure of cities that the nature and intensity of land use vary greatly from city to city and from one part of a city to another. Again, it is but a small step to recognize that a major element of factor substitution is involved in this phenomenon and to analyze models whose production functions will explain the observed factor substitution. Indeed, factor substitution is the most dramatic characteristic of urban structure. For example, the relative price of housing varies somewhat from one part of a city to another, but such variation is small compared with the variation in the relative prices of factors used to produce housing—principally land and structures. It is not unusual for land values to vary by a factor of from ten to one hundred within a distance of ten or twenty miles in a large metropolitan area. And the tremendous variation in capital-land ratios—from skyscrapers and high-rise apart-

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ments downtown to single story factories and single family homes on two-acre lots in the suburbs—is the market's response to these dramatic variations in relative factor prices.

The model developed below is intended to shed light on these and other factors. To keep the mathematics within manageable proportions, it is necessary to make significant compromises with reality. In the work that follows, two major areas of compromise can be identified. First, the demand side has been slighted almost to the point of exclusion. This has been necessary in order to focus attention on what seem to me to be the crucial factors; namely, input substitution and technology. Second, the degree of aggregation is uncomfortably high. Even with these two areas of compromise, the model is quite cumbersome. Its solution is pragmatic and inelegant.

II. *A World without Cities*

It is clear that the existence, size, and structure of cities are closely related to transportation costs. The avoidance of transportation costs is not, however, a sufficient reason for the existence of cities. Indeed, it may help in focusing ideas to state explicitly a set of assumptions—each of which finds a respectable place in important economic models—which imply that there would be no cities.

Consider a general equilibrium model in which an arbitrary number of goods is produced either as inputs or for final consumption. The only nonproduced goods are land and labor, each of which is assumed to be homogeneous. Assume that each production function has constant returns to scale and that all input and output markets are competitive. Utility functions have the usual properties and have as arguments amounts of inputs supplied and products consumed. Under these circumstances, consumers would spread themselves over the land at a uniform density to avoid bidding up the price of land above that of land available elsewhere. Adjacent to each consumer would be all the industries necessary—directly or indirectly—to satisfy the demands of that customer. Constant returns assures us that production could take place at an arbitrarily small scale without loss of efficiency. In this way, all transportation costs could be avoided without any need to agglomerate economic activity.

III. *An Abstract Description of a City*

The two assumptions in the previous section most in conflict with reality are that land is homogeneous and that production functions all have constant returns to scale. Relaxation of either is sufficient to justify the existence of cities. Reasons for relaxing them and for the alternatives to them that are employed below are discussed in the next two paragraphs.

If some land is more productive than other land, it will pay to concentrate production on the better land, thus producing a city. The location of almost all U.S. cities can be understood in terms of land heterogeneity, most having been located near cheap water transportation. There are two ways to represent this heterogeneity in formal models. One is to assume that several variables related to land enter the production functions—natural resources, topography, climate, etc.—and that these variables are available in different amounts at different sites. Another is to assume just one land input, but to assume that different sites have associated with them different efficiency parameters in production functions. For a variety of reasons, the latter representation is chosen in this paper. With this convention, I would say that Baltimore's location results from the fact that some goods—especially transportation services—can be produced more efficiently there than further inland. The limited availability of desirable land will show up as decreasing returns as the amount of land used increases, forcing resort to less and less productive land. I will summarize this assumption by saying that efficiency parameters require locational indexes.

Location theorists have identified a variety of factors that lead to "agglomeration economies." The most important and best articulated of these factors is increasing returns to scale. This leads to agglomeration, not only of the activity in question, but also of other activities vertically related to it. Among other sources of agglomeration economies, most can probably be represented approximately as scale economies, at least in an aggregative model. Provided that the notion of scale economies is interpreted broadly, so as to include indivisibilities, it is undoubtedly important in determining city sizes. There are large numbers of specialized business and consumer services for which the per business or per capita demand is so small that a large city is needed to support even a few suppliers.

It is obvious that either locational effects on efficiency parameters or increasing returns will justify the existence of a city. Furthermore, conditions of production impose a finite limit on the efficient size of the city. Suppose we consider the possibility of doubling the population of a city by doubling the height of every building. If this were feasible and if twice as many people now traveled between each pair of points as before, then it would lead to just twice the demand for transportation as before. But if transportation requires land as an input, it must use more land after the doubling of population than before. Thus, some land previously used for buildings must now be used for transportation, thus requiring new buildings at the edge of the city. But the edge of the city has now moved out, and some people must make longer trips than before, requiring more transportation inputs. Thus, a doubling of the city's population requires more than doubling transportation inputs.

For a city of sufficient size, this "diseconomy" in transportation will more than balance any economy of size resulting from increasing returns in production. Another factor that entails the same result is the fact that, as the city's population grows, efficient production of goods requires the use of somewhat more land as well as of somewhat higher structures. At least this is true of any production function that has diminishing returns to factor proportions. Consequently, as a city grows, it moves out as well as up, and this entails diseconomy in transportation resources.

It was suggested above that the exhaustion of favorable land may show up as decreasing returns to scale in production. On the other hand, it was also stated that increasing returns in production is the most important agglomeration economy. It is thus important to formulate a model that is consistent with either increasing or decreasing returns to scale and to let the data tell us which assumption is appropriate.

IV. *The Model*

A. *Production Conditions.* The model developed here is an aggregate one. It assumes that only three activities take place in the urban area.

The first activity is the production of goods. The goods production function justifies the existence of the city. The city may be located where the efficiency parameter in the production function for goods is especially favorable. The production function may have increasing or decreasing returns. If there is no effect of location on the efficiency parameter, we must have increasing returns. Otherwise, there would be no city. If there are increasing returns, it is assumed that they are available only if goods production takes place in a contiguous area. If, instead, the city exists because of a site with a favorable efficiency parameter, then goods production will take place at this site. In either case, goods production will take place in a contiguous area and assumptions to be made below will imply that this area plays the role of the central business district (CBD). The production function is assumed to be Cobb-Douglas. Formally,

$$X_1 = A_1 L_1^{\alpha_1} N_1^{\beta_1} K_1^{\gamma_1} \qquad \alpha_1 + \beta_1 + \gamma_1 = H_1 \gtrless 1 \tag{1}$$

where X_1 = total output of goods, and L_1, N_1, K_1 are total inputs of land, labor, and capital in goods production. When written as in (1), the symbols refer to the amounts of inputs and outputs in the city. When reference is made to the value of a variable at a particular distance from the city center, the dependence on distance will be indicated explicitly. Thus, $X_1(u)du$ refers to the amount of goods produced in a ring of width du centered on a circle u miles from the city center. Then

$$X_1 = \int_{\text{city}} X_1(u)\,du$$

The second activity is intracity transportation. Assumptions will be introduced below that imply that the production of housing and other activities locate in "suburbs" around the CBD in a pattern determined by their bids for land. Transportation links the CBD with these suburbs. A great deal of factor substitution is possible in transportation. At one extreme, subways use little land but much capital. At another extreme, cars use much land but rather little capital. Probably the most realistic representation would be to assume that a choice must be made among a finite number of input-output coefficients relating inputs of land, labor, and capital to output of transportation. An efficient transportation system might then require the choice of a different set of coefficients in different parts of the city. However, the need for an integrated system places limits on this choice. Investigation of an optimal transportation system within the framework of the model developed here is a major goal of this study. The present paper, however, is restricted to studying the implications for city structure of choice of a particular set of coefficients. Thus, the coefficients are assumed to be exogenous in this paper. Actually, only one such coefficient is relevant for further analysis, as will be shown below. It is the ratio between land and transportation:

$$L_2(u) = bX_2(u) \tag{2}$$

$X_2(u)du$ is the number of passenger miles of transportation produced within a ring of width du u miles from the city center, and $L_2(u)du$ is the land input in transportation in this ring.

The third activity is designated "housing." The assumption is that all commodities whose production functions have nonconstant returns to scale and whose efficiency parameters are affected by location can be aggregated into the production function for goods (activity one). Competition will force the production of all other commodities to be located adjacent to customers in order to avoid transportation costs. It is assumed that the production of such goods and of housing can be aggregated into a single production function, designated "housing."

$$X_3(u) = A_3L_3(u)^{\alpha_3}N_3(u)^{\beta_3}K_3(u)^{\gamma_3} \quad \alpha_3 + \beta_3 + \gamma_3 = 1 \tag{3}$$

Once again, the u designates inputs and outputs within a narrow ring u miles from the city center. The assumption that all commodities can be dichotomized into the two groups designated as goods and housing is an approximation. In fact, there are degrees to which conditions of production require central location. Shopping centers display sufficient increasing returns to prohibit neighborhood location, but not enough to require central location except in small towns. To introduce intermediate commodities of this sort would vastly complicate the model, since it would require each activity to be located not only with reference to distance from the center but also with reference to the distance from its neighbors.

B. *Market Conditions.* All factor markets are assumed competitive, so that each activity pays the same price for a given factor. Furthermore, the wage rate, w, and the rental rate on capital, r, are assumed to be exogenous. These are the appropriate assumptions if the city's size is to be endogenous. The city's population is determined by the number of workers it can bid for at the going wage rate. Likewise for the city's capital stock. The rental rate per acre of land u miles from the city center, $R(u)$, is endogenous.

Market conditions in the goods industry must be specified carefully. If there are increasing returns to scale, we cannot also have competitive product and factor markets. In this model it is assumed that the goods producer is a monopolist. The demand for X_1 is

$$X_1 = a_1 p_1^{-\lambda_1} \qquad \lambda_1 > 1 \tag{4}$$

where λ_1 is the constant elasticity of demand. X_1 should be thought of as an "export" good. Alternatively, a_1 could be made a function of the city's population, although that possibility has not been incorporated into the subsequent analysis. A careful limiting operation in which λ_1 goes to infinity, but a_1^{1/λ_1} remains finite, would permit perfect competition to be included as a special case of (4).

It follows from the assumptions made in this model that the CBD will be circular. It can therefore be characterized by a single number, k_o, the distance from the city center to the boundary of the CBD. In order to increase land inputs for CBD uses, land must be bid away from suburban uses. Land rent at the boundary of the CBD, $R(k_o)$, therefore determines the use of land in the CBD. CBD land users take this rent as fixed, but its value will be determined by the model. Factor demands by industry 1 are thus determined by the following marginal productivity conditions:

$$\frac{\partial(p_1X_1)}{\partial L_1} = R(k_o), \qquad \frac{\partial(p_1X_1)}{\partial N_1} = w, \qquad \frac{\partial(p_1X_1)}{\partial K_1} = r$$

Because of (4), these can be written as

$$\bar{\alpha}_1 \frac{p_1X_1}{L_1} = R(k_o), \qquad \bar{\beta}_1 \frac{p_1X_1}{N_1} = w, \qquad \bar{\gamma}_1 \frac{p_1X_1}{K_1} = r \tag{5}$$

where

$$\bar{\alpha}_1 = \epsilon_1\alpha_1, \qquad \bar{\beta}_1 = \epsilon_1\beta_1, \qquad \bar{\gamma}_1 = \epsilon_1\gamma_1, \quad \text{and} \quad \epsilon_1 = 1 - 1/\lambda_1.$$

It is assumed that actual rent paid in the CBD just absorbs any monopoly profit. Thus,

$$R_1 = \frac{p_1X_1 - wN_1 - rK_1}{L_1} = \frac{(1 - \bar{\beta}_1 - \bar{\gamma}_1)}{\bar{\alpha}_1} R(k_o) \tag{6}$$

the last equation following from (5). We must have $R_1 \geq R(k_o)$, otherwise industry 1 could not bid any land away from the surburbs. This inequality requires

$$H_1 \leq \frac{\lambda_1}{\lambda_1 - 1} \tag{7}$$

This inequality says that the greater the extent of increasing returns in goods production, the more inelastic must be goods demand in order to be able to pay the factors their marginal revenue products. It shows that the more competitive the goods market (the larger λ_1), the less the extent of increasing returns that is consistent with the model. In the limiting case, perfect competition requires constant or decreasing returns. (7) is assumed to hold in what follows.

Housing is assumed to be produced with competitive output—as well as input—markets. Thus

$$\alpha_3 \frac{p_3(u)X_3(u)}{L_3(u)} = R(u), \qquad \beta_3 \frac{p_3(u)X_3(u)}{N_3(u)} = w, \qquad \gamma_3 \frac{p_3(u)X_3(u)}{K_3(u)} = r \tag{8}$$

Using (3) and (8), we get the well-known expression for output price when markets are competitive and the production function is Cobb-Douglas.

$$\begin{aligned} p_3(u) &= \left[A_3\alpha_3^{\alpha_3}\beta_3^{\beta_3}\gamma_3^{\gamma_3}\right]^{-1} R(u)^{\alpha_3} w^{\beta_3} r^{\gamma_3} \\ &= \bar{A}_3 R(u)^{\alpha_3}, \qquad \bar{A}_3 = \left[A_3\alpha_3^{\alpha_3}\beta_3^{\beta_3}\gamma_3^{\gamma_3}\right]^{-1} w^{\beta_3} r^{\gamma_3} \end{aligned} \tag{9}$$

It is assumed that housing consumption per worker is independent of u. Although this is not strictly correct, it is justified by the fact, stated above, that variations in the proportions in which land and capital are used to produce housing are much greater from one part of a city to another than are variations in the amount of housing consumed. We can express this assumption as

$$X_3(u) = N(u)x_3 \tag{10}$$

where $N(u)$ is the number of workers resident at a distance u from the center, and x_3 is the constant per worker housing demand.

It is assumed that a fraction ρ of the workers resident at each u is employed adjacent to their residences in the suburbs. It would be better to allow this proportion to be determined by the model, and presumably the conclusion would be that ρ would increase with u. Efforts to incorporate this possibility into the model have been unsuccessful. The assumption made amounts to the assumption that a fraction ρ of the workers resident at each u are employed in housing and transportation, and a fraction $(1-\rho)$ commute to the CBD. It is assumed that a transportation system adequate to handle these CBD commuters is also adequate for all other purposes. This is an accurate assumption for radial transportation and no other form appears in the model. With this assumption, the number of passenger miles of transportation needed at each $u \geq k_o$ is proportional to the number of workers who live beyond u and who commute to the CBD. By an appropriate choice of units, the factor of proportionality can be put equal to one:

$$X_2(u) = (1 - \rho) \int_u^{k_1} N(u')du' \qquad k_o \leq u \leq k_1 \tag{11}$$

Here k_1 is the distance from city center to the outer edge of the suburbs. k_1 is endogenous. Likewise, the amount of transportation needed at a $u \leq k_o$ is proportional to the number of workers employed closer to the city center.

$$X_2(u) = \int_o^u N_1(u')du' \qquad 0 \leq u \leq k_o \tag{12}$$

This ignores the commuting demand of transportation workers. This is legitimate if commuting is by car, since the commuters are then also the transportation workers. Otherwise it is an approximation.

It is assumed that the cost per passenger mile of transportation is proportional to $R(u)$:

$$p_2(u) = aR(u) \tag{13}$$

This follows literally from (2) if it is assumed that land is the only transportation input. More realistically it is intended to reflect the fact that a major cost of intra-urban travel is the opportunity cost of time spent traveling and that travel is inevitably slower in denser, higher rent areas, even in an optimum transportation system. Although (13) is not necessarily the most realistic assumption that could be made, it greatly simplifies subsequent analysis.

A worker resident at u could decrease his transportation costs by moving in toward the city center. Equilibrium in the location of housing requires that no such move be profitable. This will be so if the change in transportation cost from a short move is just offset by an opposite change in housing cost. This assumption can be expressed by the following equation:

$$p_2(u) + p_3'(u)x_3 = 0 \tag{14}$$

where the prime designates a derivative w.r.t. u. This crucial assumption appears in several models of urban location, but its implications appear not to have been analyzed.

The final assumption concerning market conditions is that urban users must be able to bid land away from some other uses, such as agriculture, at the edge of the urban area. Thus,

$$R(k_1) = R_A \tag{15}$$

where R_A is the opportunity cost of using land for urban purposes. R_A is exogenous, and (15) provides an "initial" condition for $R(u)$.

C. *Other Conditions*. Equilibrium requires that all land be used for some purpose. Within the CBD, land is used to produce goods and transportation, and we must have

$$L_1(u) + L_2(u) = 2\pi u \qquad 0 \leq u \leq k_o \tag{16}$$

In the suburbs, land is used to produce transportation and housing, and we must have

$$L_2(u) + L_3(u) = 2\pi u \qquad k_o \leq u \leq k_1 \tag{17}$$

(16) and (17) assume that there is no obstruction to a circular city. Topographical considerations—such as lakes, rivers, and harbors—may make a city of this shape impossible. If the obstruction is shaped like a pie slice, no fundamental alteration is necessary. If an obstruction takes up $(2\pi-\theta)$ radians at each u, then 2π can be replaced by θ in (16) and (17) and wherever 2π appears subsequently. Irregular obstructions cannot be handled within this model.

The relationship that completes the model says simply that all workers must live somewhere. This can be expressed as

$$N_1 \equiv \int_0^{k_o} N_1(u)du = (1-\rho)\int_{ko}^{k_1} N(u)du \tag{18}$$

V. *Solution*

Despite the fact that the model presented in Section IV is drastically oversimplified in an economic sense, it is mathematically cumbersome. There does not seem to be any way of checking uniqueness or consistency by counting equations and unknowns, or any simple method of solution. Proceeding pragmatically and taking advantage of special properties of the model, it is, however, possible to solve it. The endogenous variables are input and output quantities and prices in the three activities, the rent of land, and the distribution of residences—all expressed as functions of u. We should also be able to derive k_o and k_1. Of greatest interest are the expressions giving the rental value of land, the allocation of land among competing uses, and the density of population, each expressed as a function of distance from the center. Exogenous are the parameters of the three production functions, parameters of the demand function for goods, prices of labor and capital, the fraction of the labor force employed in the suburbs, the demand for housing per worker, and the rental value of land for agricultural purposes.

A. *CBD*. First, consider k_o and $R(k_o)$ to be fixed. Then from (5) we get the land-labor ratio in CBD goods production. Using (2), (5), (12), and (16), we get

$$L_1(u) + \lambda R(k_o) \int_0^u L_1(u')du' = 2\pi u \qquad \lambda = \frac{b\beta_1}{w\alpha_1}$$

Differentiating once, we get a first order differential equation in $L_1(u)$. Using the initial condition $L_1(0) = 0$, the solution is

$$L_1(u) = \frac{2\pi}{\lambda R(k_o)} (1 - e^{-\lambda R(k_o)u}) \tag{19}$$

This shows that the amount of land available for production increases at a decreasing rate as one moves out from the city center, despite the fact that the total amount of land available grows proportionately to u. The reason is that the land needed for transportation at u is proportionate to the integral of $N_1(u)$ up to u, and this grows much faster than u. Substituting (19) into (16), we have

$$L_2(u) = 2\pi\left[u - \frac{1}{\lambda R(k_o)} (1 - e^{-\lambda R(k_o)u})\right] \tag{20}$$

If the city is sufficiently large, both k_o and $R(k_o)$ will be large. In that case, for large u, $L_2(u)$ is approximately $2\pi u$. This interesting result shows that, in a sufficiently large city, transportation will require nearly all the land near the edge of the CBD. But it cannot require more land than is available. That is, if CBD factor ratios are those dictated by competitive factor prices, the CBD will always be of a size such that there is enough land at the edge of the CBD to transport all those who work in the CBD.

This result also sheds an interesting light on CBD traffic congestion. Excessive congestion is not inherent in large city size. No matter how large the city there exists an allocation of CBD land that will avoid the need for increases in passenger miles of transportation per acre of CBD land allocated to transportation. Congestion comes about because of the way cities grow. As a city grows (e.g., because of an increase in A_1 or a_1), $R(k_o)$ increases. As can be seen from (20), $L_2(u)$ is an increasing function of $R(k_o)$ for every u in the CBD. This is because an increase in $R(k_o)$ entails an increase in N_1/L_1 (and in K_1/L_1), which requires that a larger amount of CBD land be devoted to transporting the increased number of CBD workers. Congestion results because the adjustment of N_1/L_1 (and K_1/L_1) is relatively quick, whereas the transfer of CBD land from goods production to transportation is relatively slow. The former adjustment takes place mostly in the private sector, whereas the latter normally requires a transfer of land from the private to the public sector.

These and subsequent results can also be used to answer the following question, although the analysis has not been carried out. Suppose that CBD land is now allocated optimally, but that the city is expected to grow. Then three possibilities exist: (1) congestion will take place; (2) land will be transferred from goods production to transportation; (3) input-output coefficients in transportation must change (e.g., a switch from automotive to mass transit). What combination of the three is most economical? As it stands, the model considers only alternative (2), and it assumes that the city starts from scratch in that the cost of using CBD land for transportation is its rental value. For an existing city, the cost of transferring CBD land from goods production to transportation is its improved value, and this is much larger than its unimproved value.

So far we have considered only the input side of goods production. (19) tells us how much land will be available at each u in the CBD if factor proportions are those dictated by competitive factor prices. Taking account of the amount of X_1 that can be sold at the profit maximizing price, we get an expression for the demand for L_1. Making use of (1), (4) and (5), we get

$$L_1 = \bar{L}_1 R(k_o)^V$$

$$\bar{L}_1 = \left[A_1 a_1^{1/(\lambda_1-1)}\left(\frac{\beta_1}{\bar{\alpha}_1 w}\right)^{\beta_1}\left(\frac{\bar{\gamma}_1}{\bar{\alpha}_1 r}\right)^{\gamma_1}\bar{\alpha}_1\frac{\lambda_1}{\lambda_1-1}\right]^{(\lambda_1-1)/(\lambda_1-H_1(\lambda_1-1))} \tag{21}$$

$$V = \frac{(\beta_1+\gamma_1)(\lambda_1-1)-\lambda_1}{\lambda_1-H_1(\lambda_1-1)}$$

It is easy to see that (7) implies $V<0$, so that the higher are CBD land values, the less CBD land is demanded for goods production. It is also easy to check that an increase in A_1 or a_1 increases X_1 and L_1, as we should expect. We will return to (19) and (21) below where we will see how they can be used to determine k_o and $R(k_o)$.

B. *Suburbs.* Primary attention will be focused on finding the functions

$R(u)$ and $N(u)$. These are the most interesting variables from the theoretical and policy points of view. In addition, once these functions have been found all the input functions can easily be found using (2), (8), (9), (10) and (11).

$R(u)$ will be derived first. Substituting from (9) and (13) into (14), we get a differential equation in $R(u)$. Using the initial condition (15), the solution is

$$\begin{aligned} R(u) &= [R_A{}^{-(1-\alpha_3)} - c(k_1 - u)]^{-1/(1-\alpha_3)} \quad k_o \leq u \leq k_1 \\ c &= (1 - \alpha_3)a(\alpha_3\bar{A}_3x_3)^{-1} \end{aligned} \tag{22}$$

It is sometimes asserted or speculated that land values fall off exponentially as one moves out from the city center. It is interesting to observe that exponential decline is a special case of (22) where $\alpha_3 = 1$. This is the special case where there is no factor substitution possible in housing; land is the only input. In general, (22) indicates a slower-than-exponential decline in land rents. Exponential decline means $R'(u)/R(u)$ is constant, whereas when $\alpha_3 < 1$, $R'(u)/R(u)$ is a decreasing function of u in (22). Another way to put this is to say that the possibility of economizing on land in housing prevents land values from rising as fast as they otherwise would as one moves in toward the city center.

Now turn to population (strictly, labor force) distribution, $N(u)$. Substituting from (2), (9), (10) and (11) in (17), we get

$$b(1 - \rho)\int_u^{k_1} N(u')du' + BR(u)^{-(1-\alpha_3)}N(u) = 2\pi u$$

$L_2 = b X_2$

Using the solution for $R(u)$ in (22), and differentiating, we get a differential equation in $N(u)$, whose solution is

$$\begin{aligned} N(u) &= c_1D^{-1}[BR_A{}^{-(1-\alpha_3)} - a(1 - \alpha_3)(k_1 - u)]^D - 2\pi D^{-1} \quad k_o \leq u \leq k_1 \\ B &= \alpha_3x_3\bar{A}_3, \qquad D = b(1 - \rho) - a(1 - \alpha_3) \end{aligned} \tag{23}$$

c_1 is an arbitrary constant of integration. Using the expression involving $L_3(u)$ in (8) and (9), (10), (15), and the initial condition $L_3(k_1) = 2\pi k_1$, c_1 can be evaluated as

$$c_1 = 2\pi k_1B^{-(1+D)}DR_A{}^{(1-\alpha_3)(1+D)} + 2\pi B^{-D}R_A{}^{D(1-\alpha_3)}$$

It is to be noted in (23) that $N(u)$ is an increasing function of u if $D > 0$, and a decreasing function of u if $D < 0$. It is not possible to specify the sign of D a priori. This is as it should be. As u increases, three things happen: (1) the total amount of land increases proportionately to u; (2) the amount of land needed for transportation decreases; (3) population per residential acre decreases. (1) and (2) tend to increase $N(u)$, whereas (3) tends to decrease $N(u)$. The net effect depends on the strengths of (1)–(3), and these are measured by the coefficients that make up D.

The behavior of population density is generally of greater interest than is $N(u)$. Net density[1] is $N(u)/L_3(u)$, whereas gross density is $N(u)/2\pi u$. Using the first expression in (8) and (9), (10) and (22) net density is given by

$$\frac{N(u)}{L_3(u)} = [BR_A{}^{-(1-\alpha_3)} - (1 - \alpha_3)a(k_1 - u)]^{-1} \tag{24}$$

That is, the reciprocal of net density is linear in u. Colin Clark [1] has argued that population density falls off exponentially in all cities and at all times. Unfortunately, Clark does not make it clear whether he is using net or gross density and presents his evidence in a way that is difficult to evaluate. (No R^2's or significance tests are given, and there is no statement as to how the excluded CBD was determined.) Nevertheless, it is worth noting that no special case of (24) yields an exponential density function. Nor does any special case of this model yield an exponential gross density function.

The form of (24) makes empirical estimation and testing particularly easy. The reciprocal of net population density can be regressed on distance from city center. Furthermore, although the constant term in this regression (which depends on k_1) will vary from city to city, the coefficient of u should be the same for cities of different size. This provides an extremely simple partial test of the model.

Unfortunately, the effects of parameter changes on population density are not easy to ascertain within this model. Clark [1] asserts that an increase in transportation cost will increase population density near the center and decrease it further out. Within the model presented here, an increase in a represents an autonomous increase in transportation costs. The direct effect of an increase in a is as Clark states. But a change in a will also affect k_o and k_1, and the net effect is difficult to ascertain. Among the questions one would like to answer, the most interesting would concern the effects of taxes and subsidies on the transportation system.

C. *Determination of k_o and k_1.* All the solutions presented in the last two subsections, and solutions for other variables not presented, contain the two values k_o and k_1 in addition to the autonomous parameters of the model. Equations to determine these values can be specified as follows.

k_o must be such that the land available for goods production in the CBD equals the land that can be profitably employed in CBD goods

[1] Since land used for goods produced in the suburbs is included in $L_3(u)$, the measure of net density used here is not "as net" as the ratio of population to land used for housing.

production. The former is an integral of (19).

$$L_1 \equiv \int_o^{k_o} L_1(u)du = \frac{2\pi}{\lambda R(k_o)}\left[k_o - \frac{1}{\lambda R(k_o)}(1 - e^{-\lambda R(k_o)k_o})\right] \quad (25)$$

The latter is given by (21). Equating the two gives

$$\overline{L}_1 R(k_o)^V = \frac{2\pi}{\lambda R(k_0)}\left[k_o - \frac{1}{\lambda R(k_0)}(1 - e^{-\lambda R(k_o)k_o})\right] \quad (26)$$

Upon inserting the expression for $R(k_o)$ from (22), (26) becomes an equation involving only k_o and k_1 among the endogenous variables.

k_o and k_1 must also be such as to provide enough land for housing in the suburbs to house the workers who work there and those who work in the CBD. (18) ensures this. Upon substituting the solution for N_1 and the solution for $N(u)$, this too becomes an equation involving only k_o and k_1. Thus, (18) and (26) provide two equations for the two unknowns k_o and k_1. Although some progress has been made with approximations, the equations appear too complicated to learn much from them without resorting to numerical methods.

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