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Beyond the power law – a new approach to analyze city size distributions

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Abstract

This work proposes a new approach to analyze the city size distribution (CSD). We present a general equation for the rank size logarithmic plot, with a new positive exponent α. When $\alpha = 1$, the Pareto distribution is yielded; when $\alpha \neq 1$, the log of the curves exhibits a concave distribution. We studied the CSDs of 41 cases in 35 countries (in several countries we examined cities and metropolitan areas or agglomerations) in order to apply our new equation. We determined accurately the exponent α for 31 cases. In 18 cases we received $\alpha = 1$, in one case $\alpha < 1$, and in 12 cases $\alpha > 1$. However, for the other cases, either the distributions were not homogeneous, or the data exhibited significant fluctuations which precluded a good determination of the exponent α. Based on this analysis, we developed a series of models (based on the models of town growth of Gabaix and of Blank and Solomon) in order to describe the different CSDs. The results of these models include power laws as well as cases that are represented by concave distributions on a logarithmic plot of the rank size.

Keywords: City size distribution; Power laws

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1. Introduction

Auerbach was the first to propose the idea that a Pareto distribution (or a power law) represents city size distribution (CSD) accurately. A Pareto distribution states that the number of cities with a size larger than a given size A is proportional to A^{-P} where P is the Pareto exponent. Ever since Auerbach proposed this theory, this perception has become a widely held opinion among scholars in a variety of disciplines. Zipf (1941) provided an empirical analysis which suggested that the size distribution of cities can be approximated by a Pareto distribution with an exponent of unity. This rule is also known as "Zipf's Law". Extensive literature that was developed subsequently focused on applying the Pareto distribution in general and Zipf's law in particular to CSDs in different countries.

Several works in the last two decades have shown, using empirical analysis, that Zipf's law is not always tenable. CSD was accepted as a Pareto distribution with an exponent seldom equal to one. In these cases the name "rank size rule" was adopted (Alperovitch, 1984; Kamecke, 1990; Rosen & Resnick, 1980; Soo, 2002; Urzúa, 2000). These works and others (Alperovitch, 1993; Brakman, Garretsen, Marrewijk, & Van den Berg, 1999; Gabaix, 1999; Krugman, 1996; Reed, 2002) have tried to explain the Pareto exponent by means of economic, geographic, and social processes.

Over time some doubts were raised regarding the validity of the Pareto distribution. Husing (1990) and Alperovitch and Deutsch (1995) argued that not only Zipf's law, but the Pareto distribution as well were not always supported by the empirical data. Cameron (1990) has shown that a two-parameter Weibull distribution fits the empirical data better than the Pareto distribution. More recently, Laherrere and Sornette (1998) proposed the use of a stretched exponential distribution. Some scholars (Rosen & Resnick, 1980; Soo, 2002) have suggested that the rank size rule may be used as a first approximation, despite the fact that it is not always observed. To summarize, work that has been done on this subject thus far intended to find a unique framework that would permit a general study of CSD. Nevertheless, a definite conclusion has not yet been reached.

The aim of this work is twofold: first, to propose a new approach to analyze the city (or cities agglomeration) size distributions. We present a new approach for the study of CSD. Instead of searching for a unique framework or for a mathematical function which fits the CSDs of all the countries around the world, we seek a specific function for each individual country. Then, we group these functions into classes. To emphasize the importance of this new approach, we display the logarithmic plot of size versus rank of the CSDs of India and China (Fig. 1). There is no need for any mathematical tool to realize that India's distribution fits a power law (see the straight line with a slope of 0.82), while China's distribution differs from it. We argue that the power law distribution cannot be used even as an approximation for China. Our second goal is to support our analysis by introducing a computer model which yields all kinds of CSDs that correspond to all of the observed cases of CSDs. Previous models focused on the Pareto distribution only. The evolution of system of cities is a dynamic process, thus, we suggest that different types of CSDs could be related to different phases in the evolution of the system of cities (agglomerations). To support this hypothesis, we show that by introducing dynamics into a model, it yields not only the Pareto distribution, but also the entire spectrum of CSDs.

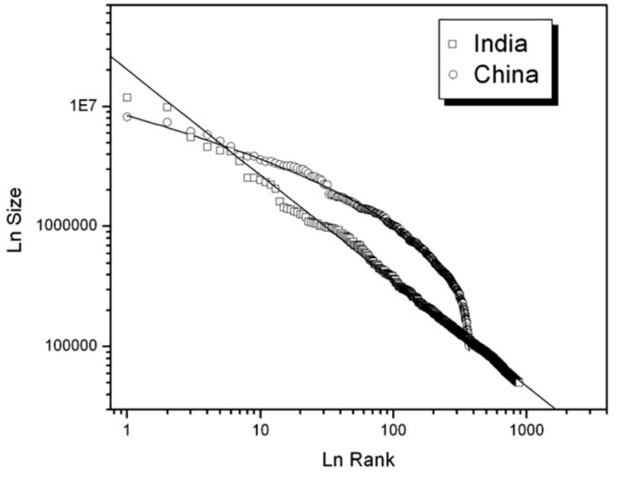

Fig. 1. The rank size plots on logarithmic scales of the cities in India (1991) and China (1990) present the differences between the two distributions. The "Size" axis represents the population divided by 1000.

2. The specific functions of CSDs

In this study we searched for the best mathematical description for CSDs using a sample of 41 countries all over the world. The logarithmic plot of size versus rank was used as the simplest representation of the data. We chose countries with no less than 18 cities or agglomerations larger than 100,000 inhabitants (http://www.citypopulation.de/). We did not distinguish between cities and agglomerations in this work, as our intention was to show that there is no such thing as a universal distribution (meaning one function that can describe all CSDs around the world). The choice of 18 cities is somewhat random, but it is based on our assumption that a distribution has a mathematical significance only for a large enough number of cities. In this work we do not discuss the issue of the minimum size that a city needs to be in order to be taken into consideration in the distribution. Nevertheless, we acknowledge the importance of this issue. For the present work, we considered cities with a population of no less than 100,000 inhabitants. This number was chosen as to exclude the lower tail of CSDs in countries with a very large number of cities (e.g. India, China). However, we are aware that choosing a smaller minimum size of cities thus giving a larger distribution may yield a better answer for the uniqueness of the function.

We were guided by two essential principles: first, some distributions could not be described by a unique expression since they were piecewise.[1] These distributions were excluded from our analysis as we assume they corresponded to non-homogeneous distributions (see further discussion in Section 4). Second, we considered the phenomenon of the primate city, which is the largest and most important city in a country. A primate city is more than twice the size of the next largest city and works as the core of the country in terms of economic, cultural, and political aspects (Jefferson, 1939). Thus, primate cities cannot automatically be considered as an integral part of CSDs. In this work, we

[1] A function $f(x)$ is defined piecewise, if $f(x)$ is given by different expressions on various intervals (e.g. a step function).

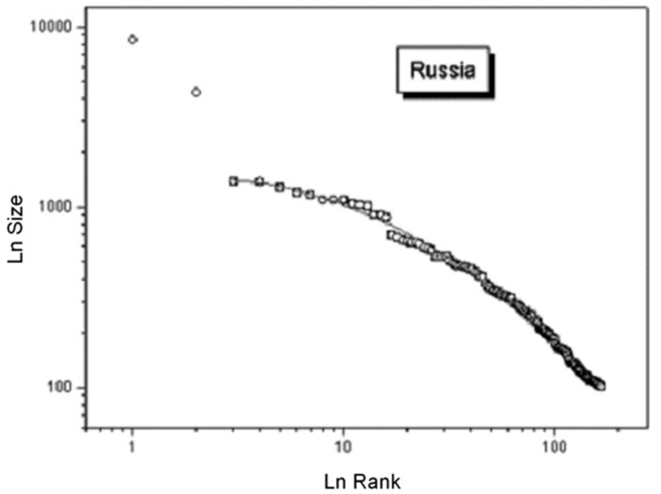

Fig. 2. The rank size distribution of Russia's cities (1994).

examined for each case, whether the first one or two largest cities can be considered as a homogenous part of the CSD, or rather they diverge from the distribution of the rest of the cities and should not be included in the analysis. In the case of Russia, for example, the two largest cities (Moscow and St Petersburg) have been the capital city of Russia at different times (Fig. 2). It is clear that they are not a homogenous part of the CSD, and it is necessary to exclude them from the analysis in order to get a continuous distribution.

To examine whether a specific function fits a specific distribution we used the software package of Origin®. Our analysis was based on two stages: in the first stage we considered the possibility of a power law as the best fit to any CSD. We determined whether to accept this fit following two criteria: the first was the coefficient of determination (R^2), which indicates the goodness of the fit (we chose $R^2 \geqslant 0.97$). The second criterion was the plot of the differences between the observed data and the values of the linear equation. We rejected the linear equation when the differences presented a systematic deviation. If a straight line was excluded, we could not consider the distribution as a Pareto one and thus proceeded to the second stage. In the second stage, we examined concave functions.[2] These functions could be described as 'parabola-like' curves with a concave distribution. After examining the entire sample of 41 cases we found two general formulas, which seemed different at first glace, to describe all cases of homogenous CSDs. The first one is (y represents ln(size), where x represents ln(rank), a and b are parameters, and α is a new positive exponent)

$$y = y_0 - a(b + x)^{\alpha} \tag{1}$$

In Eq. (1) α is always larger than or equal to 1. Although α is not necessary an integer, when equal to 1 and 2 Eq. (1) presents unique characteristics; when $\alpha = 1$ the curve corresponds to a linear equation. When $\alpha = 2$ the curve described by Eq. (1) corresponds to a parabola with a symmetry axis parallel to the y-axis. When $\alpha > 1$ this curve

[2] A function $f(x)$ is said to be a concave function if its second derivative is negative (i.e. downward curvature).

corresponds to a parabola-like function (with a symmetry axis parallel to the y-axis). The second formula that describes the CSDs is

$$y = y_0 + a(b - x)^{\alpha} \tag{2}$$

In Eq. (2) α is smaller than 1. The curve described by Eq. (2) corresponds to a parabola-like function with a symmetry axis parallel to the x-axis. When $\alpha = 0.5$ the curve is an exact parabola (with a symmetry axis parallel to the x-axis). The case $\alpha = 1$ yields (again) a linear equation but with different values of the parameters a and b.

To combine Eqs. (1) and (2) into a unique formula we used a step function $H(\alpha)$. If $\alpha < 1$ the function $H(\alpha) = -1$ if $\alpha \geqslant 1$ the function $H(\alpha) = 1$. The general formula is thus:

$$y = y_0 - H(\alpha)a[b + H(\alpha)x]^{\alpha}. \tag{3}$$

3. The meaning of the new exponent

In order to understand the influence of the new exponent α on the CSD, we have analyzed the properties of Eq. (3). For that, we examined the function $N(A)$ which is a cumulative function that presents cities with size larger than or equal to A. In Eqs. (1)–(3) $y = \ln A$ and $x = \ln R$ (R represents rank). If the rank size distribution is given by $A(R)$ then:

$$N(A) = R(A) \tag{4}$$

If $\alpha > 1$, then $R_c = \exp(b)$ and $A_0 = \exp(y_0)$, therefore:

$$R = \frac{1}{R_c} \exp\left[\left(\frac{1}{a}\right)^{\frac{1}{\alpha}} \left(\ln \frac{A_0}{A}\right)^{\frac{1}{\alpha}}\right] \tag{5}$$

It is clear that $A \leqslant A_0$, i.e. no city has a size larger than A_0. Therefore, A_0 might represent the largest city. The distribution function $P(A)$ giving the number $P(A)\,\mathrm{d}A$ of cities with sizes between A and $A + \mathrm{d}A$ is given by $(-\mathrm{d}R/\mathrm{d}A)$. This is a function with a steep slope, as $P(A)$ approaches zero for $A \to A_0$ (Fig. 3a). In other words, the behavior of $P(A)$ in this case can be considered as pseudo-divergent.

If $\alpha < 1$, R_c and A_0 have the same meaning as above and the result is

$$R = R_c \exp\left[-\left(\frac{1}{a}\right)^{\frac{1}{\alpha}} \left(\ln \frac{A}{A_0}\right)^{\frac{1}{\alpha}}\right] \tag{6}$$

In this case $A \geqslant A_0$, and A_0 might represent the smallest city. However the distribution $P(A)$ has a particular property; it exhibits a maximum for the cities with the smallest sizes (Fig. 3b) which is a surprising result. It is largely accepted that in a city size distribution, $P(A)$ is a regularly decreasing function. Nevertheless, an example for a situation with a maximum point was found in the empirical data (see the case of China in Fig. 4).

Finally, from Eqs. (5) or (6), we recover the well known result for $\alpha = 1$. The function $N(A)$ is a Pareto distribution:

$$N(A) = R_t \left(\frac{A_m}{A}\right)^{\frac{1}{a}} \tag{7}$$

where R_t is the largest rank and A_m is the smallest size.

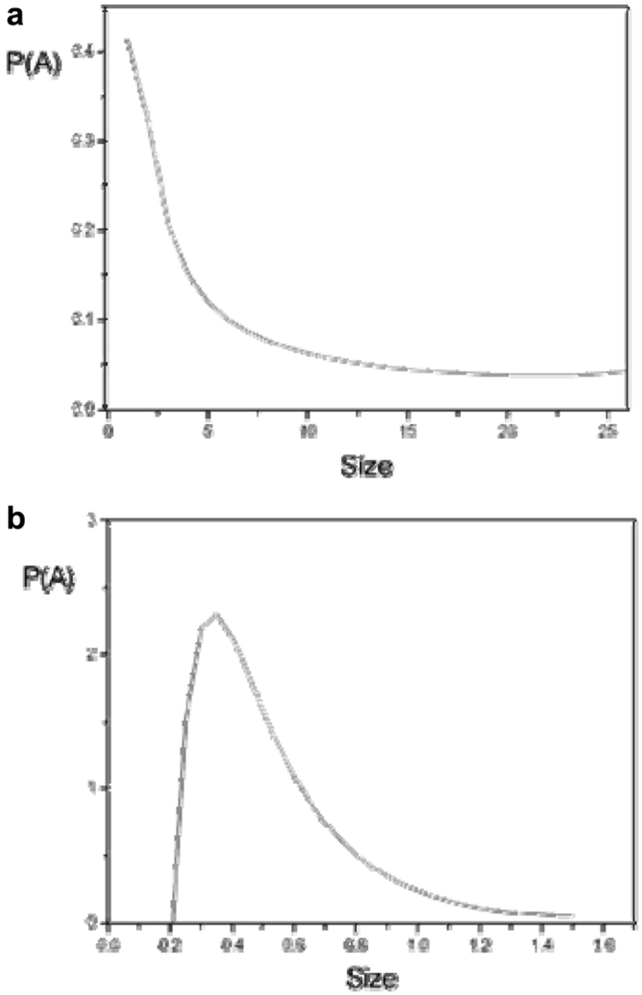

Fig. 3. The distribution function $P(A)$ for: (a) $\alpha > 1$ and (b) $\alpha < 1$.

4. The results of the analysis

We fitted the entire sample of 41 cases with Eq. (3) using the Origin® software and achieved the following results:

1. For 17 countries, we got Pareto distributions ($\alpha = 1$) with exponents different from unity (see Table 1). However, in France the Pareto distribution was reached after excluding the primate city. The exponent in this case was remarkably smaller than unity. We found a Pareto exponent equal to unity only in 6 of these cases (with an error of 10%), while for the rest of the cases it varied from 0.61 (France) to 1.25 (Canada-Urban areas).
2. For 13 cases, we found values of α different from unity (see Table 2). This values varied from 0.41 (China) to 2.59 (Tanzania). In China, Germany-Agglomerations, and in the

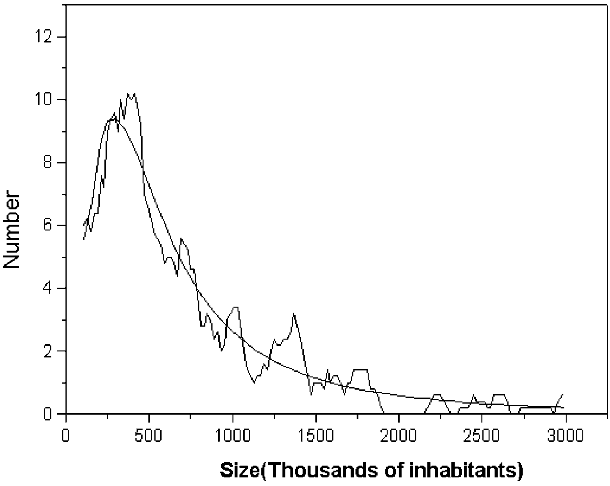

Fig. 4. The probability distribution of China.

Table 1
Distributions that fit a power law

	Country	Year	a
1	France-cities	1999	0.61 ± 0.01
2	Japan	2000	0.72 ± 0.01
3	Italy	2001	0.74 ± 0.01
4	USA-cities	2003	0.74 ± 0.01
5	Brazil	2003	0.82 ± 0.01
6	Spain	2003	0.86 ± 0.01
7	India-cities	2001	0.88 ± 0.01
8	India-agg.	2001	0.91 ± 0.01
9	Iran	1996	0.95 ± 0.01
10	Nigeria	1991	0.96 ± 0.01
11	Turkey	2000	0.96 ± 0.01
12	South Korea	2002	1.01 ± 0.01
13	Pakistan	1998	1.02 ± 0.01
14	Marocco	2004	1.11 ± 0.01
15	Colombia	2003	1.12 ± 0.01
16	Kenya	1999	1.16 ± 0.02
17	Canada-Urban areas	2001	1.25 ± 0.01

USA-metropolitan areas we considered the entire data while for the rest of the countries we had to exclude the primate cities in order to achieve this result (i.e., Ethiopia, Poland, France-Agglomerations, Congo, Ukraine, Argentina, Thailand, Mexico, Uganda, and Tanzania). In Table 2, there are several cases where $b = 0$. In fact, a better fit was obtained with very small values of b (order of 0.001), but as the differences were very small, we decided to adopt the value $b = 0$. This means that the rank size logarithmic curve exhibits a null slope for $x = 0$.

3. For 11 cases (even after the exclusion of the primate cities), we could not determine a unique function that fit the data. In some of these cases we observed non-homogenous

Table 2
Distributions that fit Eqs. (1) and (2) with values of α different to unity

	Country	Year	*a*	*b*	α
1	China	1990	2.21 ± 0.05	5.92 ± 0.01	0.41 ± 0.01
2	Germany-agg.	2003	0.55 ± 0.02	0	1.36 ± 0.02
3	Ethiopia	2004	0.35 ± 0.04	0	1.40 ± 0.01
4	Poland	2002	0.33 ± 0.01	0	1.45 ± 0.01
5	France-agg.	1999	0.32 ± 0.01	0	1.52 ± 0.02
6	Congo	1984	0.42 ± 0.04	0	1.55 ± 0.06
7	USA-metropolitans	2003	0.28 ± 0.01	0	1.63 ± 0.01
8	Ukraine	2001	0.25 ± 0.03	0	1.69 ± 0.06
9	Argentina	2001	0.14 ± 0.01	0.07 ± 0.10	2.00 ± 0.01
10	Thailand	2000	0.05 ± 0.01	3.02 ± 0.42	2.00 ± 0.01
11	Mexico	2000	0.06 ± 0.01	0	2.51 ± 0.03
12	Uganda	1999	0.06 ± 0.01	0	2.51 ± 0.07
13	Tanzania	2002	0.10 ± 0.02	0	2.59 ± 0.16

groups of distributions (Bangladesh – cities and agglomerations, Egypt, South Africa, the Philippines, and the UK), and in others – the distributions could have been interpreted in more than one way (Russia, Germany's cities, Indonesia, Algeria, and Vietnam). Since we concluded that there is more than one function that describes the CSD, these cases do not contradict our conclusion.

5. Discussing the new analysis of the CSD

This new analysis of the CSD introduces us to several issues. The presence of the primate cities in the analysis of CSD is essential. The basic (and very often implicit) hypothesis of the entire work in this field is that a mathematical description is a precise indication of a homogeneous group of cities. This is the reason behind several models (Gabaix, 1999; Reed, 2002) that regard cities of a specific country as forming a homogeneous ensemble. In this study, the exclusion of primate cities was made on the basis of the empirical data alone. To justify the exclusion of the primate cities from the data, it is important to consider them as cities with unique development and characteristics. For example, in countries with a centralized political system such as France or Russia, the capital cities play a special role and it is understandable that they became primate cities. We believe that it is necessary to investigate this issue further on the basis of historical, economic, and political changes. Excluding the primate cities when necessary, we reduced the analysis of the distributions to three options: (1) a Pareto distribution, (2) a concave distribution, and (3) a division of the distribution into two sub-distributions. Several proposed models (Gabaix, 1999; Krugman, 1996) focus exclusively on the possibility of recovering the Pareto distribution. However, recent models (Batty, 2004; Sembolini, 2001) suggest other possibilities.

In several cases in our analysis, we decided to divide the distribution into two sub-distributions, even when a fit with a high R^2 was obtained. Each of these sub-distributions fit a different function. Indonesia is a good example for such a case (Fig. 5). Although the fit to a straight line yielded $R^2 = 0.98$, a visual inspection strongly suggested that a division of the distribution into two segments would be more appropriate (note the discontinuity in the graph).

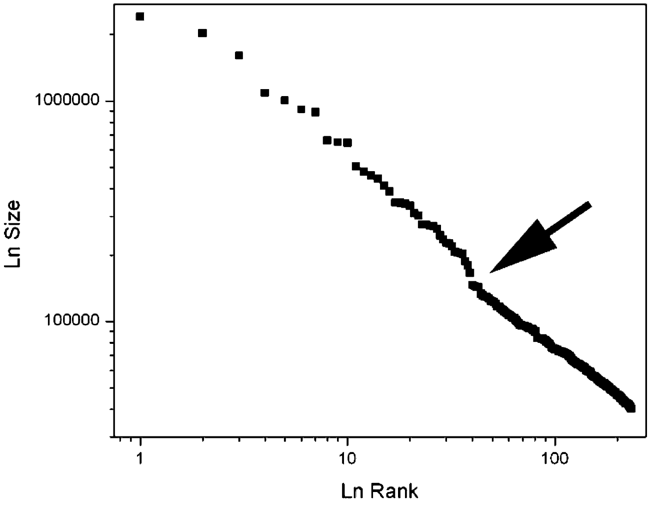

Fig. 5. The rank size distribution of Indonesia's cities (2000).

Romania is another good example of the inability to find only one function that fit the data well (Fig. 6). After the primate city of Romania was excluded, the best analysis was to divide the data into two functions. The case of two sub-distributions is analogous to the case of the primate cities; it is obvious that it corresponds to a non-homogeneous group of towns and that a thorough analysis is necessary. To understand the necessity of dividing Romania's CSD into two sections, one needs to look into its history (http://www.zum.de/whkmla/histatlas/balkans/haxrumania.html). Fig. 7 presents the evolution of the frontiers of Romania from 1500 until today. Located in the middle of three empires (the Austrian Empire, the Ottoman Empire, and the Russian Empire), different regions of the country

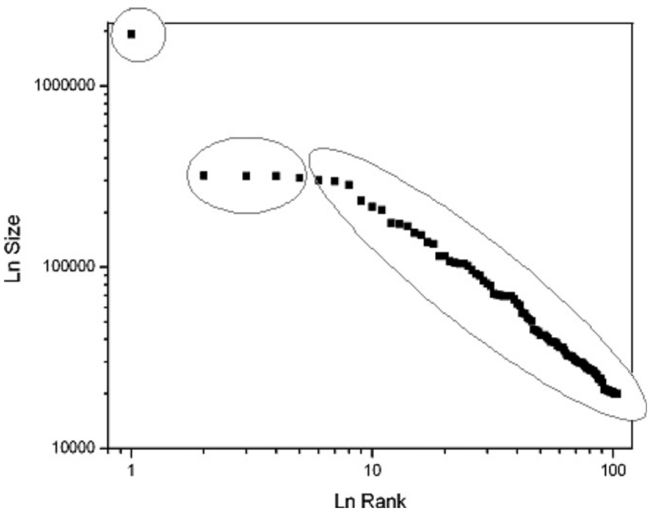

Fig. 6. The rank distribution of Romania's cities.

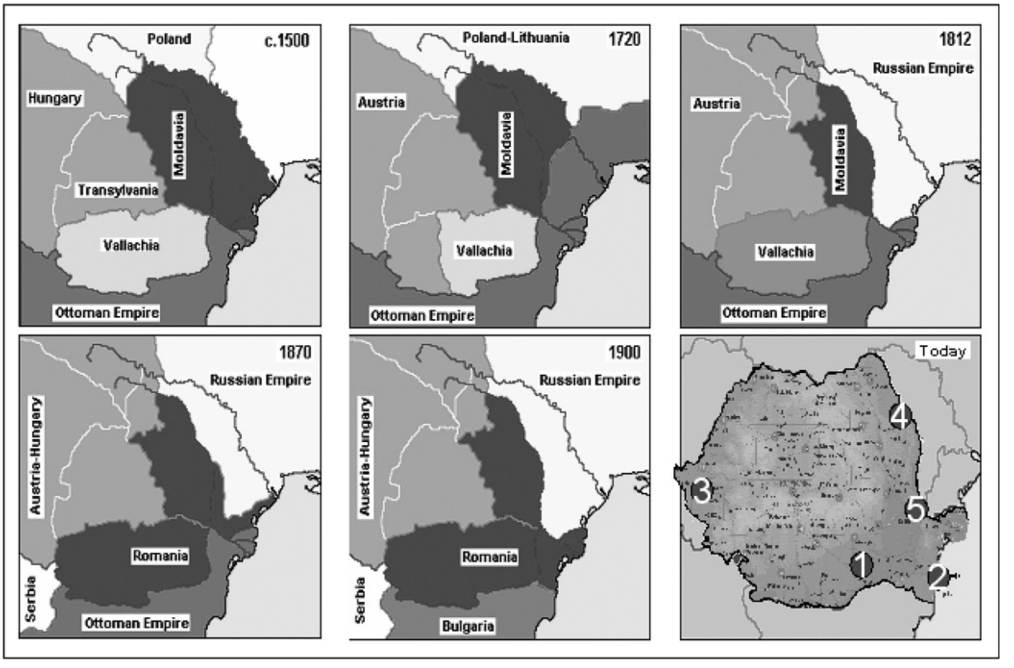

Fig. 7. The changes in the borders of Romania between 1500 and today.

were governed by different empires throughout the years. During the 19th century, two of the main regions of Romania had frontiers with one of the two empires: Moldavia with the Russian Empire, and Walachia with the Ottoman Empire. Transylvania, the third main region of Romania, was governed by the Austrian Empire. In fact, Romania was united only at the beginning of the 20th century. The biggest cities of Romania were not part

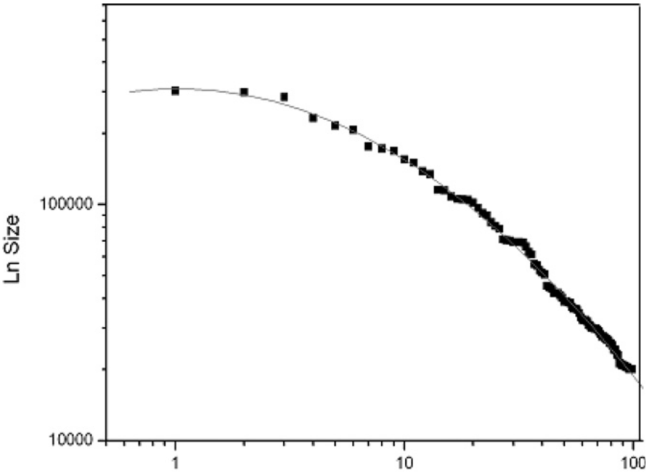

Fig. 8. The rank distribution of Romania's cities after the primate city and additional four cities were removed.

of a single system of cities and therefore they cannot conform to the distribution of the rest of the cities. When the first five cities of Romania (the primate city Bucharest, three cities which are located at the periphery: Iasi, Timisoara, Galati, and one city that is located in Transylvania: Cluj-Napoca) were excluded from the distribution, the new logarithmic plot of log(A) versus log(r), yielded a parabola (Fig. 8).

To summarize, the analysis of CSD yielded results that fit Eq. (3) with exponent α smaller than, equal to, or larger than 1. The Pareto distribution, which is accepted in the literature as a good first approximation for CSD, was rejected for the majority of the 41 examined cases. In the following section, we present a computer model which yields all type of observed CSDs.

6. The model

Having analyzed real data of CSDs, we examined whether models that simulate growth processes of cities yield CSDs that correspond to Eq. (3) with exponent α smaller than, equal to, or larger than 1. We believe that such a result could strengthen the validity of our analysis.

In this section, we introduce a few versions of a simulation model. This model is based on previous ideas, however, previous work that has been done on the CSD focused on power law distributions only. In this work, we examine all other observed distributions as well.

In realty, urban systems are dynamic entities that change through time, which affects their size distribution. The boundaries of these systems can be affected by events such as wars and occupation (e.g. Romania), and each entity within the system can be affected by various events such as agglomeration with neighboring cities (e.g. the separated districts of New York City) or destruction by occupation (e.g. Assur, 614 b.c.). Thus, we introduce dynamics into the model and show that by doing so, it yields (for different length of iterations) CSDs that correspond to Eq. (3) with exponent α smaller than, equal to, or larger than 1.

Based on the above, we suggest that different types of CSDs might be related to different phases in the evolution of systems of cities. Additionally, the qualitative difference between size distributions of cities and agglomerations in the same country (e.g. USA and France) could be explained by the different phases in these systems' evolution.

Our model is based on the model of Blank and Solomon (2000), which was inspired by Gabaix (1999), with a few modifications. The model calculates the growth of N cities with population $P_i(t)$ for city i. Each step of the program follows time t. At step $t+1$ a city is picked at random and its population can either increase or decrease by:

$$P_i(t+1) = \gamma P_i(t) \tag{8}$$

where γ is a random variable, uniformly distributed between 0.9 and 1.13. As the mean value of γ is 1.015, the overall population increases. We looked for two values, one is slightly smaller and one larger than 1 with a mean value larger than 1. These values were chosen as a result of experiments, to enable the growth process to be mainly positive. After N steps on average, all the cities are chosen and consequently their populations are modified.

We present three versions of this model: in the first version (A), the only constraint on the growth of the cities is caused by the pre-determined choice of a unique distribution for the random variable γ. The second version (B) is the exact model of Gabaix (1999), except for the fact that it is solved by the computer program. It states that the smallest city has a population $P_{\min}$ which is correlated with the mean size of the cities' population P_{mean}: $P_{\min} = cP_{\text{mean}}$. c is a parameter smaller than 1, which was set in our realization to vary from 0.05 to 0.5.

In models A and B, we used a constant number of cities $N = 100$, based on the mean number of cities within several countries. However, this specific number is not significant (Blank & Solomon, 2000). The initial populations of the cities were chosen as equal to 1 or 10, and the number of steps varied from 10^4 to $4 * 10^5$.

In the third version (C), N can change during the realization of the model following two processes. The first is the creation of new cities, similar to the models of Simon (1955) and Steindl (1965). The second is the disappearance of a city if its population decreases below a given size. The use of these two processes is the novelty of the model of Blank and Solomon (2000), who claimed that these two processes together are necessary to reach the CSD that corresponds to Zipf's law.

In model C (see Fig. 9), there were 100 initial cities with a population equal to 1. The initial population of each city, created in the following steps, was equal to 1 as well. When the population dropped below 1, the city disappeared (i.e. it was excluded from the model). In fact, in order to maintain its existence, the population of a new city has to take off immediately after its creation.

It is possible to understand the action of both processes as a random creation of cities, contrary to the models of Simon (1955) and Steindl (1965), where the rate of the creation of new of cities is constant. We examined two methods for the creation of new cities. In the first method we used a constant rate of the creation of cities (i.e. one city was created every K steps during the entire running time T). In the second method, we used an increasing rate of the creation of the cities (i.e. the frequency of the creation of a new city increased after a determined number of steps within T). We believe that the use of this method is

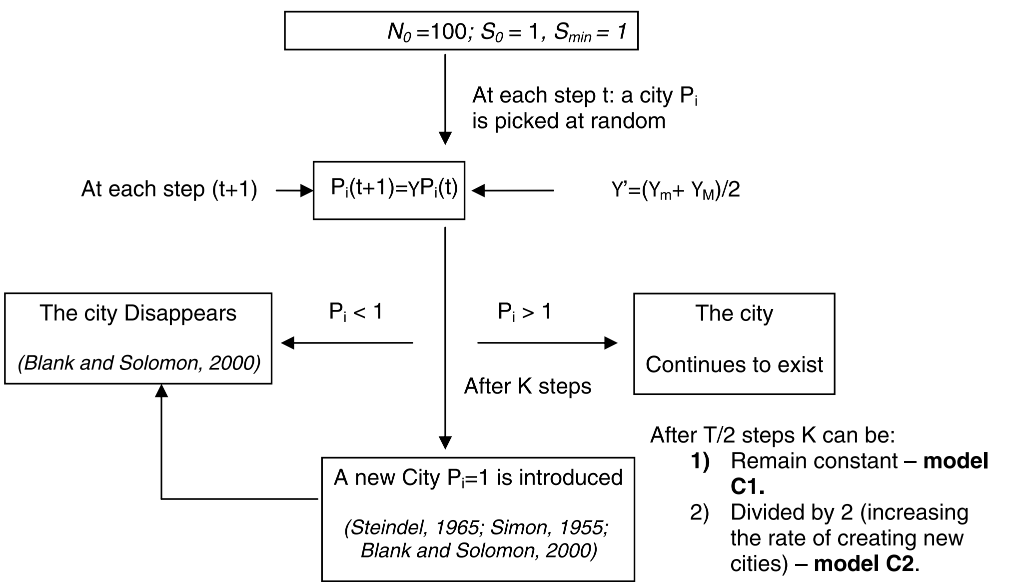

Fig. 9. A flow chart for model C.

simpler than the method used by Blank and Solomon. In their model, the number of cities increases at each step following:

$$N(t+1) - N(t) = Q[P_{\text{tot}}(t+1) - P_{\text{tot}}(t)] \quad (9)$$

where Q is a constant and P_{tot} represents the total population of the entire cities. This rule implies that the rate of the creation of a new city increases with time (similarly to our second method). However, as we will further show, the results of our model were very similar to those of Blank and Solomon, even though we did not use Eq. (9). We conclude that the use of Eq. (9) is not necessary in order to reach Zipf's law. Contrary to their claim that their results were insensitive to their choice of a constant Q, we achieved different results with different rates of creation of new cities.

7. The results of the model

7.1. Model A

Fig. 10 presents the results of the model for $T = 2 * 10^4$ and $T = 10^5$. These results are not innovative as they were previously presented by Blank and Solomon (2000) and by Batty (2004). Nevertheless, as we will discuss further, these results are interesting and contribute to our final conclusions. The fit of the results of this model with Eq. (3) yielded values of the new exponent α between 0.3 and 0.4. In model A the change of the values of T was interlaced within the values of parameter y_0. In other words, the CSD presents the same type of curves regardless of the value of T.

7.2. Model B

We present the log–log rank size plot of the CSDs for the following cases:

(1) $c = 0.1$, $T = 20{,}000$, and $T = 100{,}000$ (Fig. 11).
(2) $c = 0.5$, $T = 10{,}000$, and $T = 50{,}000$ (Fig. 12).

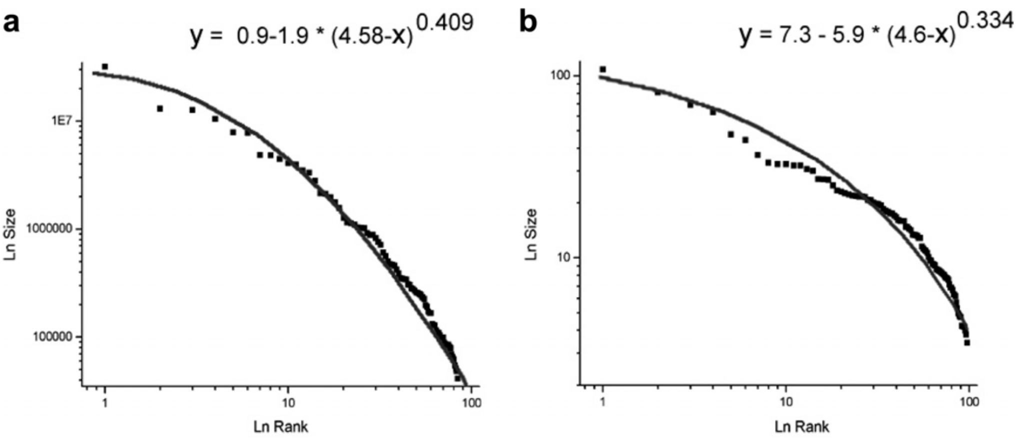

Fig. 10. Results of model A for: (a) $T = 20{,}000$ and (b) $T = 100{,}000$.

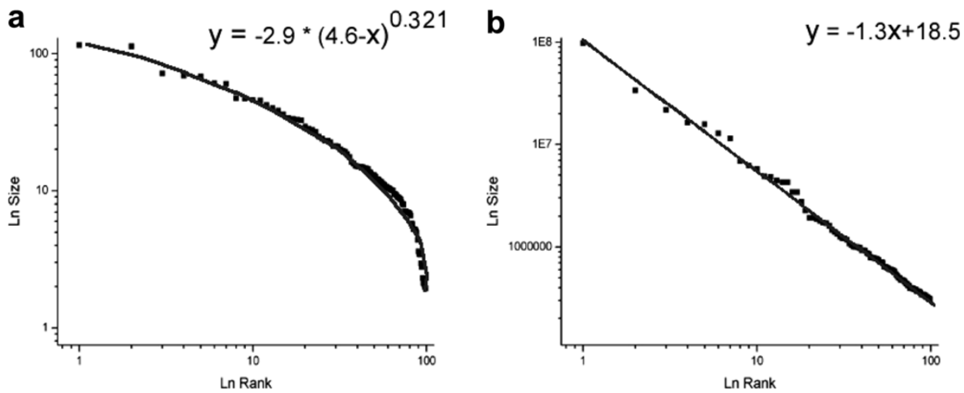

Fig. 11. The log–log rank size plot of the CSDs for model B with: (a) $c = 0.1$, $T = 20{,}000$ and (b) $c = 0.1$, $T = 100{,}000$.

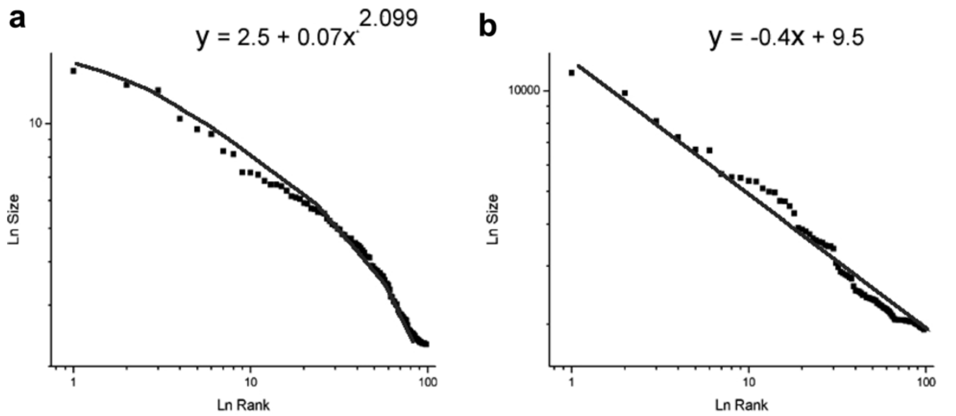

Fig. 12. The log–log rank size plot of the CSDs for model B with: (a) $c = 0.5$, $T = 10{,}000$ and (b) $c = 0.5$, $T = 50{,}000$.

Fig. 11 shows that for low values of T (short periods) the results are very similar to the ones of model A. However, as the values of T increased (longer periods) the CSDs moved closer to a linear equation with an exponent a dependent on c. Figs. 11 and 12 show that the results of cases with small values of T fit Eq. (3) with $\alpha \neq 1$; while the larger value of T fit a linear equation, i.e., Eq. (3) with $\alpha = 1$. Fig. 13 presents the variation of the exponent a (Pareto distribution) versus the parameter c. The last result is very similar to that of Blank and Solomon except that our model proves that in order to reach a linear equation, it is necessary to run the model for at least a minimum time $T_{\min}$. The values of $T_{\min}$ decrease when the values of c increase. Fig. 13 also indicates that the values of a fall between 0.8 and 1.2 when the value of c is approximately 0.15, which is a reasonable value.

7.3. Model C

As mentioned earlier, this model is characterized by a variable number of cities. In model C1 the rate K^{-1} of creation of new cities is constant (Fig. 9). In Figs. 14 and 15

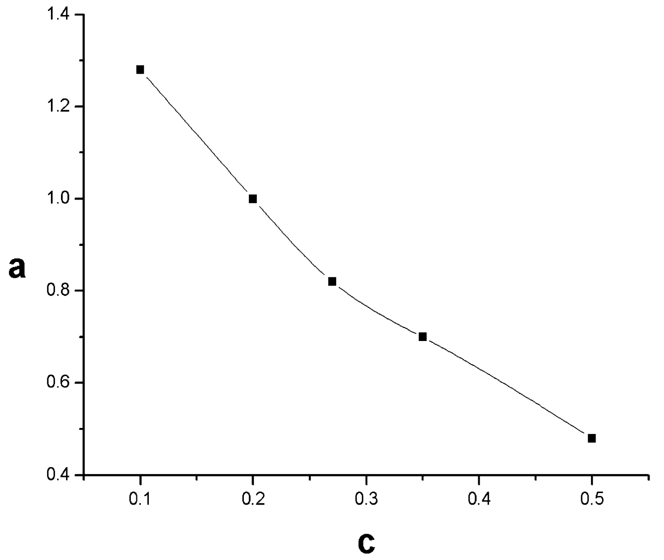

Fig. 13. The variation of exponent a (Pareto distribution) versus parameter c (the smoothing is a based on spline analysis).

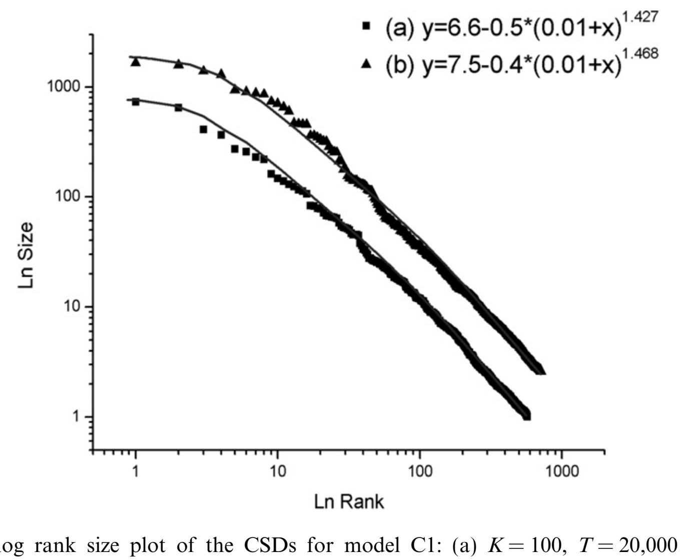

Fig. 14. The log–log rank size plot of the CSDs for model C1: (a) $K = 100$, $T = 20{,}000$ and (b) $K = 100$, $T = 50{,}000$.

the log–log rank size plot of the CSDs for two rates ($K = 25$, and $K = 100$) are presented for different times T. The shape of the curves is hardly dependent on the values of T for $K \geqslant 100$ (Fig. 14), and the curves can be easily fitted by Eq. (3) with $\alpha \neq 1$. However, for smaller values of K and high enough values of T, the curves reach the linear equation (Fig. 15). The exponent a equals 0.74 for $K = 50$, and 0.44 for $K = 25$. Yet it is impossible

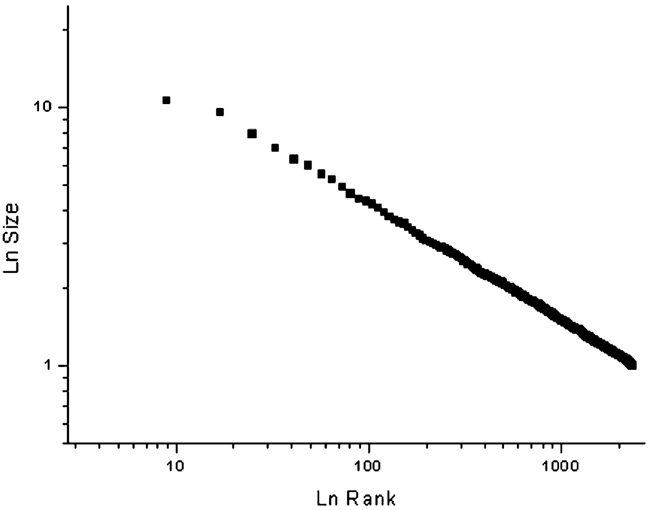

Fig. 15. The log–log rank size plot of the CSDs for model C1 with $K = 25$, $T = 20{,}000$.

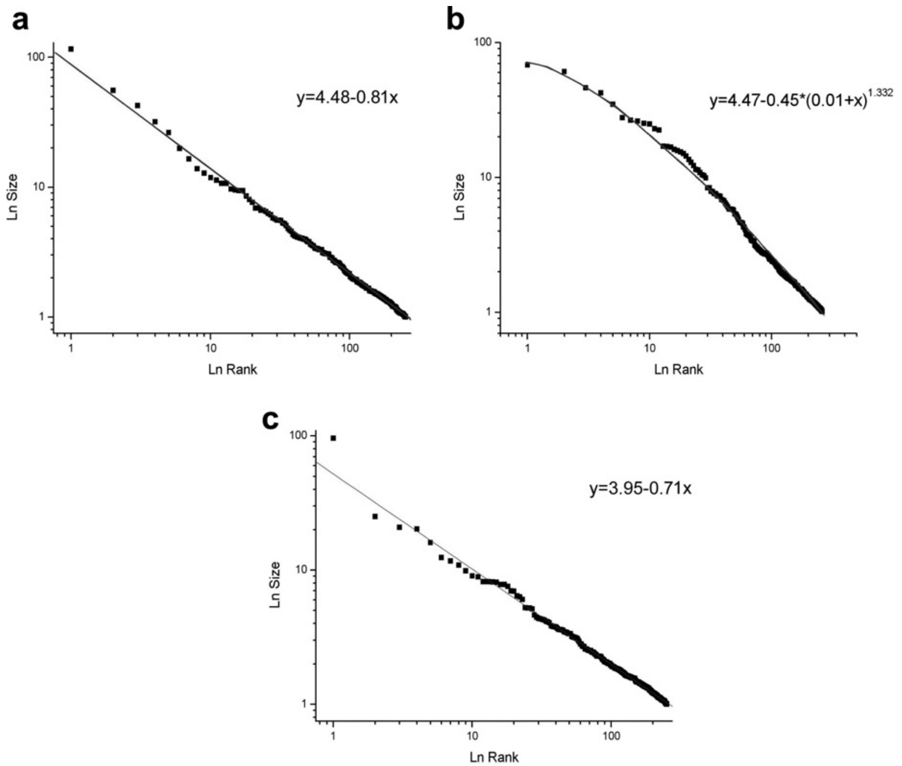

Fig. 16. The log–log rank size plot of the CSDs for model C2 with $K = 100$, $T = 50{,}000$.

to reach higher values of a by increasing the values of K, as a linear equation cannot be reached. When increasing the values of T the number of the cities increases significantly.

In model C2 the rate K^{-1} is constant up to $T/2$, and is multiplied by two afterwards i.e. from $T/2$ to T (Fig. 9). Model C2 surprised us since for the same initial parameters its results were qualitatively different in different realizations. In other words, depending on the realization, different types of CSDs were observed: (1) a straight line (Fig. 16a); (2) a concave distribution with $\alpha > 1$ (Fig. 16b); and (3) a distribution that has a primate city and the distribution of the rest of the cities fits Eq. (1) with $\alpha \geqslant 1$ (Fig. 16c).

For 20 realizations (with $K = 100$), we got 11 cases of a downward curvature and nine cases of a straight line (i.e., approximately half of the realizations yielded a straight line). In this last case the values of the exponent a varied between 0.84 and 0.98. When the rate K^{-1} was changed and the Pareto exponent (in cases where $\alpha = 1$) was measured, we found that a equaled approximately 1 for $K = 110$ and approximately 1.4 for $K = 200$. These results cover all observed CSD.

8. Discussing the results of the model

As mentioned, we assume the type of the CSD might be related to the phase of evolution of the system of cities (agglomerations). In order to examine this hypothesis, we conducted a qualitative comparison of its results with the CSD of a specific country as it varied with time. The qualitative comparison between model C2 and the CSD of the USA is presented in Fig. 17. It can be seen that the results of the simulation model fit very well the empirical data. This implies that models for CSD must be dynamic and be compared with real data over time.

At the present stage, it is difficult to conduct a precise quantitative comparison with real cases of CSDs. However, we can say that the presented model represent all the known cases of CSDs. All the models (model A which is independent of time, model B, and model C) yield the result of $\alpha < 1$. Both models B and C yield the Pareto distribution with $0.5 \leqslant a \geqslant 1.3$ that corresponds to the observed values of a in the empirical analysis. The case $\alpha > 1$ can also be obtained by models B or C. At this stage, we cannot relate the values of the parameters of the different versions of the model to specific socio-economic

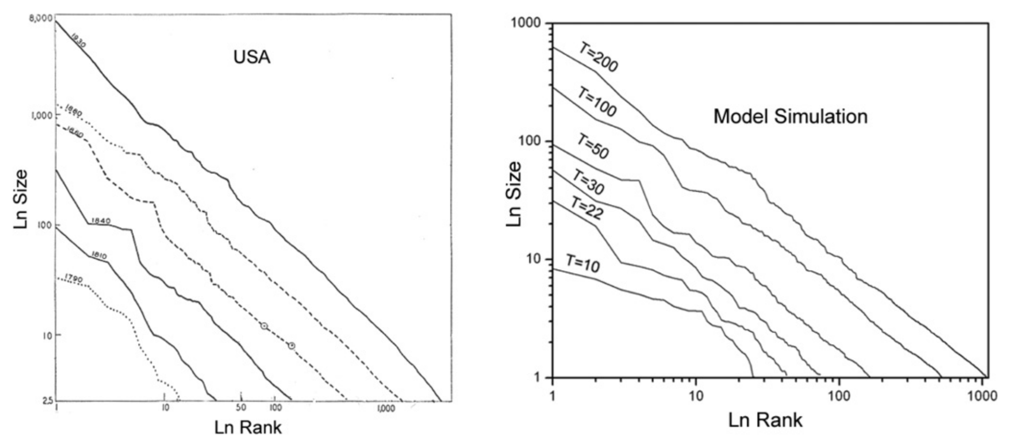

Fig. 17. A qualitative comparison between the model C2 and the CSD of the USA (T represents the total number of steps in thousands).

processes. Furthermore, in order to determine which model should be applied to a certain distribution, a thorough analysis of the country's history must be conducted. Such an analysis must include socio-economic trends and procedures that occurred in the country during the studied period. Based on such an analysis one could verify whether the concept of a minimal size of the population is relevant and if the number of cities has varied in the time interval that is considered for the study.

The results of model C2 are the most surprising ones as this model yielded $\alpha > 1$ or $\alpha = 1$ for the same conditions. This means that these two cases are not fundamentally different. A close examination of Fig. 16 reveals that the distinction between the different realizations appeared only in the large cities. However, for the majority of the cities, there was no qualitative difference. The behavior of the large cities seems chaotic. This can be related to the fact that large cities are very sensitive to historic events and alternations (which are mimicked in the model by the series of random numbers of the realization) and behave individually in relation to the other cities. Therefore, it might be justified to analyze CSDs at two scales; the distribution of large cities and the distribution of the rest of the cities.

The principal ingredient of model C2 is the variation of the rate K^{-1} during the growth. The rate of the creation of new cities corresponds to an increasing function. In comparison to the model of Blank and Solomon, our model is much simpler yet it presents all observed types of the CSDs.

9. Conclusion

In this paper, we presented a new approach to analyze the CSD, based on an empirical analysis of 41 cases. This analysis yielded results that fit Eq. (3) with exponent α smaller than, equal to, or larger than 1. The Pareto distribution, which is represented by $\alpha = 1$, was rejected for the majority of the examined cases. Based on these result we suggest that one should be very cautious about relating the values of a to economic processes (e.g. Rosen & Resnick, 1980).

Next, we examined whether models that simulate growth processes of cities also correspond to Eq. (3) with different values of α. We presented the results of computer simulations of the CSD, based on the models of Gabaix (1999) and of Blank and Solomon (2000). The versions of the model developed here are much simpler than their former ones and at the same time exhibit the complexity of the CSDs. As the evolution of systems of cities is a dynamic process, we introduced dynamics into the presented model. As a result, it yielded not only the Pareto distribution but also all observed cases of CSDs (i.e. $\alpha \neq 1$). We suggest that different types of CSDs (meaning $\alpha = 1$, $\alpha > 1$, and $\alpha < 1$) might be related to the dynamic evolution of urban system, i.e. to different phases in the evolution of the system of cities (or agglomerations of cities) in different countries. This assumption is intuitive at this stage, and was tested only on the USA, thus, further work is necessary in order to validate it.

The innovation of versions C1 and C2 of this model is the creation and disappearance of cities. It is remarkable since it indicates an apparently chaotic behavior of large cities and suggests that a more exact analysis can be conducted if the large cities are excluded from the distribution. Based on the presented analysis and models, it seems that not a, but α and K are the significant parameters and their values could be related to socio-economic processes. Further work should focus on the relation between these parameters and the actual processes that lead to their specific values.

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